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Optimal Asset Allocation of a Pension Fund: Does The Fear of Regret Matter?

By Oyakhilome IBHAGUI [†]

Abstract. In this paper, which presents a simplified behavioral finance model, we incorporate regret into the decision-making process of a pension fund and derive the optimal asset allocation of a final-wealth-maximizing pension fund in the accumulation and decumulation phases. We find that the optimal allocation must be congruent in both phases if and only if the pension fund is upside regret averse. In particular, our results suggest that allocation to risky assets must increase through time in the accumulation and decumulation phases so that the pension fund can realize gains from any upsides in the risky asset market, thereby maximizing final wealth and limiting the feeling of regret ex-post. Although decisions in both phases are congruent, we find that the optimal asset allocation generally depends on wealth levels. This evidence implies that separate management of the accumulation and decumulation phases of a pension fund decreases available wealth levels and is not an optimal strategy.
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1. Introduction

The major contribution of this paper lies in the use of regret theory to analyze the optimal asset allocation of a pension fund that aims to maximize the expected *modified* utility of its final wealth. Unlike in the standard expected utility framework wherein a pension fund independently considers only the investment choices it makes and performs expected utility maximization based on these choices, without recourse to other potential choices that could have been made, regret theory gives room for the pension fund to account for its favoured investment choices as well as other feasible investment choices that could be made. In essence, the pension fund experiences regret if the outcome of its investment choices is worse than the outcome of at least one of the forgone investment choices, and it rejoices if otherwise. Thus, because of the possibility of future regret, the objective function -the expected modified utility of the pension fund's final wealth- is set up in such a way as to incorporate a function that captures feelings of potential regret. This way, we are able to develop a set-up aimed at examining the extent to which the anticipation of future regret influences current investment choices and optimal asset allocation of a pension fund. The presence of a regret function distinguishes our framework, the regret theory framework, from traditional expected utility framework and serves as a novel contribution to the literature.

Accessible literature on asset allocation problems and optimal portfolio strategies for pension funds largely neglects regret theory and widely favors traditional expected utility maximization. In addition to the many documented limitations and violations of the traditional expected utility theory so elegantly

† Department of Economics, D3-06 Keynes College, University of Kent, Canterbury, CT2 7NP, UK.
☎. +00 + 44 (0)1227824319
✉. oi50@kent.ac.uk

demonstrated in the behavioral economics literature, our major discontent with the theory is that it assumes economic agents consider each outcome independently and disregard other possible outcomes. This can be interpreted to mean that pension fund managers, for example, care only about the investment choices they make. However, in practice, fund managers do experience a feeling of regret whenever forgone investment choices yield better returns ex-post than the choices they made ex-ante. This causes them to incorporate the possibility of regret in their subsequent investment decisions and asset allocation. As a simple illustration, consider a fund manager who can receive a $5 return on investment for each dollar invested in the debt capital market and either a $7.5 or $1.5 return on investment for each dollar invested in the equity capital market. If he takes a huge position in the equity capital market and finally receives a $1.5 return on investment for each dollar invested, he almost surely will experience a feeling of regret for getting less than he could have gotten had he taken most positions in the debt capital market. Next time, this experience will shape his investment decisions and therefore make him averse to regret, and this aversion will induce him to incorporate regret into his decision-making process. Despite this intuitive perspective on regret, to the best of our knowledge, nowhere has this common human behavioral tendency been incorporated into the investment decision making process to analyze the optimal asset allocation of a pension fund. It is therefore in this area that this paper fills a void in the literature.

Generally speaking, the concept of regret, developed by Bell, Loomes & Sugden (1982), is intuitive and proposes a normative theory of choice under uncertainty that explains many observed violations of the axioms upon which the traditional expected utility theory is built. Regret is a cognitively mediated emotion of pain and anger when people observe ex-poste that they took a bad decision ex-ante and could have taken an alternative decision with a better outcome. In capital markets, people experience regret when their investments give a worse performance than an alternative investment they could have easily chosen a priori. This, for instance, is in contrast with disappointment, which is experienced when a negative outcome happens relative to prior expectations. Regret, which is a powerful negative emotion, is strongly associated with a feeling of responsibility for a choice made and is known to influence decision-making under uncertainty. It involves the regret/rejoice that a person feels when he gets outcome x instead of outcome y. The theory assumes people are rational but base their decisions not only on expected payoffs or utility but also on expected regret, so that they try to anticipate future regret and consistently incorporate it into their investment decision making process. The incorporation of regret yields a modified utility and people reach their investment decisions by maximizing the expected value of this modified utility. This makes regret theory suitable for analyzing the optimal asset allocation of a pension fund.

The anticipation of future regret is so strong that it forces even Harry Markowitz to relook his very own Nobel winning asset allocation theory when confronted with a financial decision on his pension plan. His quote: '*I should have computed the historical covariance of the asset classes and drawn an efficient frontier. Instead I visualized my grief if the stock market went way up and I wasn't in it—or if it went way down and I was completely in it. My intention was to minimize future regret, so I split my pension scheme contributions 50-50 between bonds and equities.*' 'Harry Markowitz, as quoted in Zweig (1998), America's top pension fund', Money, 27, page 114. This gives further support and credibility to the claim that regret does influence optimal investment decision of a pension fund. Anticipation of future experience of regret may lead individuals to make certain decisions that contrast with expected utility paradigm. This assertion will be investigated in the context of the optimal asset allocation of a pension fund in this paper.

Meanwhile, unlike other fund managers, the case of pension funds requires the introduction of two characteristics: (i) the different behaviors of the managed funds during the accumulation (Ac) and decumulation (Dc) phases, and (ii) mortality risk. In addition, we must take cognizance of regret risk because we aim to work in a regret theoretic framework. So, this paper considers three dimensions of risk: traditional risk (volatility of final wealth), regret risk and mortality risk. To the best of our knowledge, no work on optimal asset allocation has simultaneously considered these risks. The only work, at least to our knowledge, which considers these risks in asset allocation theory, does not consider them simultaneously. For instance, Bajeux-Besnainou & Jordan (2001) consider only volatility risk, Michenaud & Solnik (2008) consider volatility risk and regret risk and Battocchio, Menoncin & Scaillet (2007) consider volatility risk and mortality risk. While the intuition of applying regret theory to asset allocation is not new, this is the first time that a formal regret theoretic approach is applied to the optimal asset allocation of a pension fund facing the aforementioned risks.

As we have motivated above, regret is a major factor when making investment choices because institutional investors, more often than not, care about their choices relative to other strategies they could have employed. Although evidence favoring the influence of regret on decision-making exists instinctively, it is surprising that the theory has caught only little attention in the field of finance. In the few available studies, Muermann, Mitchell & Volkman (2006) apply regret theory to asset allocation in defined contribution pension schemes. They find that an investor who takes regret into account will hold more risky assets (stocks) when the equity premium is low but less risky assets when the equity premium is high. Braun & Muermann (2003) apply regret theory to demand for insurance. Dodonova & Khoroshilov (2005) apply a pseudo regret theory to asset pricing. Heybati, Rahnamay & Moosavi (2011) apply regret theory to portfolio optimization. Michenaud & Solnik (2008) study currency hedging techniques for foreign assets in a regret theoretic framework and derive some interesting implications for long and short hedging positions when a foreign currency appreciates or depreciates ex-post. However, all these models offer approximate explicit investment rules outside the context of a pension fund.

In this paper, instead, we provide explicit optimal solutions for investment rules within the context of a pension fund which manages employees' contributions towards retirement. Our methodology allows the derivation of approximate closed-form solutions for optimal investment choices available to a pension fund. In particular, during the active years of the employees, the fund wealth increases because of the contributions that the employees make towards retirement while, after retirement, the fund wealth decreases because of the pension payments that the pension fund makes to the retired employees. Following Battocchio, Menoncin & Scaillet (2007), we suppose that a representative employee has no other choice at the retirement date than to receive a pension until the death time τ, which is assumed to be a random variable. The pension fund then maximizes the expected modified utility of its final wealth, in anticipation of future regret. In our model, the contribution and pension rates are constant and linked by a feasibility condition that guarantees the convenience of both the pension fund and the representative employee to amicably enter the pension contract. We argue why this feasibility condition must hold and derive its approximate closed-form expression under the assumption that the death time τ follows a log-logistic distribution. We emphasize that our result is quite different from the closed-form expressions obtained under the assumption of both a Gompertz-Makeham and a Weibull distributed death time τ as in Battocchio, Menoncin & Scaillet (2003; 2007). We also remark that the motivation for our choice of distribution for the death time τ stems from the fact that death-survival analyses for a random death time are best done under the assumption of a log-logistic distribution.

Furthermore, we consider that a pension fund does not only manage retirement funds preretirement when contributions towards future pensions are made, but also manages the remaining wealth postretirement when pensions are being paid. Therefore, since management of the remaining wealth postretirement is also the duty of a pension fund, we set up the required optimization problem for the optimal asset allocation during the entire life of the representative subscriber in such a way that the final date of the optimization problem coincides with the death time of the subscriber. After solving this optimization problem in a regret theoretic framework, we find that the optimal portfolio compositions of the pension fund are identical during the accumulation and decumulation phases. In particular, we show that the amount of wealth invested in the risky-asset class increases through time during the accumulation phase and also increases through time after the retirement date, during the decumulation phase. We claim that this behavior is borne out of regret aversion. Regret averse pension funds would increasingly retain their positions in the risky-asset class for fear of the potential regret of missing out on any boom or upside in the risky-asset market.

To summarize, in addition to other important results, our major contribution in this paper is systematic. We integrate regret into a well-defined objective function and this allows us to derive optimal investment strategies that reflect the risk and regret aversion of a pension fund. We find that the optimal asset allocation must be congruent in both phases if and only if the pension fund is upside regret averse. In particular, our results suggest that allocation to risky assets must increase through time in both accumulation and decumulation phases, so that the pension fund can realize large gains from the upside potential of the risky-asset market, thereby maximizing final wealth and limiting the feeling of regret ex-post. Although decisions in both phases are congruent, we find that the optimal asset allocation generally depends on wealth levels. This evidence implies that separate management of the accumulation and decumulation phases of a pension fund decreases available wealth levels and is not a robust strategy.

The rest of the paper flows as follows. Section 2 explains some important concepts that will aid the understanding of other ideas in subsequent sections, presents and discusses the financial model, and elaborates on the computation of feasibility condition on contribution and pension rates when the death time τ follows a log-logistic distribution. Section 3 presents the regret theoretic framework for the pension fund maximization problem, discusses regret theory and its importance for decision making under uncertainty and describes our modeling framework. Section 4 develops the objective function, applies regret theory to a pension fund which seeks to maximize its expected modified utility of final wealth, and computes the optimal allocation rule. Section 5 discusses the theoretical results in relation to the effective management of a pension fund and concludes with directions for future research.

2. The Financial Markets Pension Fund Model

We follow Brennan, Schwartz & Lagnado (1997) and Battocchio, Menoncin & Scaillet (2007) and consider a financial market with one risky-asset class (common stock) and one riskless-asset class (T-bills) having rates of return $\frac{dS}{S}$ and $\frac{dG}{G}$ respectively, and where r is a constant term interest rate. If we assume also that the dividend δ on common stocks is constant and influences returns on the risky-asset, then the joint price process follows,

$$\begin{cases} \frac{dS}{S} = \mu_S dt + \sigma_S dz_S, \quad S(t_0) = S_0 \\ \frac{dG}{G} = r dt, \quad G(t_0) = G_0 \\ dr = \mu_r dt + \sigma_r dz_r = 0, \ r \epsilon \mathbb{R} \\ d\delta = \mu_\delta dt + \sigma_\delta dz_\delta = 0, \ \delta \epsilon \mathbb{R} \end{cases} \tag{1}$$

where the parameters μ_i, σ_i $(i = r, \delta, S)$ are at most functions of the variables r, δ, S, and dz_i are increments due to Weiner process. S_0 and G_0 are deterministic, positive and represent the initial prices of the risky and riskless-asset classes respectively, while S and G are their prices at time $t > 0$. It should be noted that the dependence of the return $\frac{dS}{S}$ on r and δ is completely straightforward. Indeed, $\frac{dS}{S}$ depends on r and δ through the dependence of at least one of μ_S and σ_S on at most r, δ, S. This process may be estimated and tested statistically to ascertain the level of statistical significance of the effect of the state variables on stock returns, but that is not the focus of this paper.

2.1. The Contributions and Pensions Payments

If $U(t)$ denotes the total amount of contributions to the fund and $V(t)$ denotes the total amount of pensions paid by the fund, then $U(t)$ and $V(t)$ follow the ordinary linear differential equations

$$dU(t) = udt,\ t_0 \leq t < T \tag{2a}$$
$$dV(t) = vdt, T \leq t < \tau \tag{2b}$$

where $u > 0$ and $v > 0$ are constant and do not vary with time and both equations are valid for the death time τ occurring some periods after retirement[1] T, i.e. $\tau > T$. The pensions are paid until the death time of the subscriber and do not depend on the investment performance of the fund.

2.2. The Feasibility Condition

This is the condition that has to be satisfied before the subscriber and pension fund enter a pension contract in the first place. For this reason, the pension fund cannot freely dictate the contributions and pensions while the subscriber cannot solely dictate the pensions. The contributions and pensions cannot be chosen separately. The subscriber and the pension fund have to reach an agreement on the contributions and pensions simultaneously.

When the subscriber enters the fund, he anticipates that the expected present value of all pensions cannot be lower than the expected present value of all contributions. Similarly, the pension fund formalizes the contract with the subscriber when it is convinced that the expected present value of all pensions cannot be more than the expected present value of all contributions.

Money enters and leaves the pension fund according to the rate,

$$m(t) = \frac{dU(t)}{dt}\mathbb{I}_{t<T} - \frac{dV(t)}{dt}\mathbb{I}_{t\geq T} \tag{3a}$$

or

$$m(t) = u\mathbb{I}_{t<T} - v(1 - \mathbb{I}_{t<T}),$$

where

$$\mathbb{I}_{t<T} = \begin{cases} 1, & \text{if } t < T \\ 0, & \text{if } t \geq T \end{cases}$$

If u represents the constant contribution rate to the pension fund and v represents the constant pension rate paid to a representative subscriber, then the feasibility condition holds for the pair (u, v), $u > 0$ and $v > 0$ if

$\mathbb{E}\left[\int_0^\tau m(t)e^{-rt}dt\right] = 0$, and the resulting expression for the feasibility condition is

$$\frac{u}{v} = -1 + \frac{1-\mathbb{E}[e^{-r\tau}]}{1-\mathbb{E}[e^{-r\tau}\mathbb{I}_{\tau<T}]-e^{rT}\,\mathbb{P}(\tau\geq T)}, \tag{3b}$$

[1] We agree that the death time τ may plausibly happen before the retirement date T, but we do not model this scenario in this paper. Nonetheless, we are grateful to an anonymous referee who graciously drew our attention to this.

where τ is the random death time. The proof is in Appendix 1 and the computation of the closed form approximations for the feasibility condition is in Appendix 2.

3. Pension Fund Maximization Problem in a Regret Theoretic Framework

In this section we approach the problem of a pension fund from a regret/rejoicing point of view. Since pension funds are largely connected with collecting, pooling and investing funds contributed by sponsors and beneficiaries in order to provide means for the beneficiaries to accumulate savings over their active working years so as to finance their consumption needs in retirement, we shall assume that the main problem of a pension fund is to maximize the expected modified utility of its final wealth at the death time of its subscribers.

Contrary to the traditional choiceless utility function which is defined on outcomes of actual investment choices that a pension fund makes, the modified utility function does not only consider outcomes of actual investment choices, but it also includes a comparison of these outcomes with the outcomes of other investment choices that a pension fund could have made, but has not made. This makes it possible to incorporate regret into the modified utility function for the purpose of decision making. Regret-conscious pension funds do not only care about the expected return and volatility of their invested funds, but they also care about the deviations of the outcomes of their actual choices from the outcomes of their forgone choices. So, they face both volatility risk and regret risk. Volatility risk is linked to deviations of the invested fund's return from its expected value. Regret risk is the risk that the pension funds are going to experience a feeling of regret in the future. That is, the risk that the outcomes of their actual investment choices will be worse than the outcomes of their forgone investment choices. There is also mortality risk, which is the risk of death of a subscriber and it enters the problem through the feasibility condition.

In our work, we suppose that a pension fund can make two choices. The first is to invest/allocate a strictly positive amount $0 < \alpha < 1$ to a risky-asset class and the remaining to a riskless-asset class while the second is to allocate nothing $\alpha = 0$ to the risky-asset class and allocate all to the riskless-asset class. We suppose that the pension fund is unsure of the performance of the risky-asset class, but always knows beforehand the performance of the riskless-asset class, so that this performance serves as a benchmark with which the pension fund compares the outcome of its actual investment strategy. The pension fund is assumed to experience a feeling of regret or rejoicing on the outcome of its choice. If we suppose that the pension fund makes or prefers the first choice (of taking a huge position in the risky-asset class) knowing full well that it may later regret its action or decision if it so happens that the outcome (performance) of the second choice (of taking no position in the risky-asset class) turns out much better than the outcome of his first choice, how would its optimal allocation choice/decision be today in order to maximize the expected modified utility of its final wealth, improve its feeling of rejoicing and lessen its feeling of regret that may occur in the future/ex-post after it has exited it position to evaluate its proceeds? This is one of the questions we shall answer in this part of our research.

3.1. The Maximization Problem Setup

As we have motivated in the previous section, regret theory rests on two fundamental assumptions. The first is that agents experience the sensations of regret and rejoicing, and the second is that agents try to anticipate and take account of these ex-post sensations when making ex-ante decisions under uncertainty. The modified utility function is therefore defined over the ex-post (final) outcomes of choices and rational investors would make choices ex-ante by maximizing the expected value of this modified utility. This allows agents to take the anticipation of regret into account in an axiomatic fashion. The modified utility function is not only defined over the outcome of the choice an agent makes, but it also includes a comparison with the outcome of another choice that could have been made in the same state of the world.

If we define the expected value of the modified utility of a rational regret averse agent faced with two choices by

$$\mathbb{E}\big(\psi(x,y)\big) = \mathbb{E}\Big(U(x) + f\big(U(x) - U(y)\big)\Big) \tag{4a}$$

then the agent will seek to make a choice ex-ante that will give a final outcome x which will maximize his expected modified utility, i.e.

$$\max_{x} \mathbb{E}\big(\psi(x,y)\big) = \nabla$$

with

$$\nabla = \max_x \mathbb{E}\Big(U(x) + f\big(U(x) - U(y)\big)\Big).$$

It is important to note that if agents experience no regret or rejoicing at all, or if the regret function f is linear, then the above formulation collapses to the conventional expected utility paradigm. Details on the properties of f are included in Appendix 3.

Now that we have discussed the relevant aspects of regret theory that will aid our work on pension funds, we will proceed to how we can apply it in a pension fund context. Therefore, in what follows, we shall study the investment behavior of a pension fund in a regret theoretic framework.

3.2. The Managed Wealth of the Pension Fund

In the preceding section, we stated that the pension fund is assumed to have two choices—either it invests a strictly positive amount in the risky-asset class or it invests nothing in the risky-asset class. We further suppose that the pension fund does not prefer the choice of taking no position in the risky-asset class. This is the choice or decision of the pension fund. If $\hat{\theta}(t)$ represents the number of units of the risky-asset class and $\emptyset(t)$ represents the number of units of the riskless-asset class, then the apportioned wealth to the risky-asset class is $\hat{\theta}(t)S(t)$ and the allotted wealth to the riskless-asst class is $\emptyset(t)G(t)$. Therefore, the total wealth process $W(t)$ of the pension fund, which we shall call the outcome of its choice of taking a positive position in the risky-asset class, evolves according to

$$W(t) = \hat{\theta}(t)S(t) + \emptyset(t)G(t) \tag{5a}$$

The associated stochastic differential equation is

$$dW = \hat{\theta}dS + \emptyset dG + (S + dS)d\hat{\theta} + Gd\emptyset \tag{5b}$$

Introducing the self-financing condition suggests that changes in portfolio composition are only due to variations in the prices of assets constituting the portfolio. Therefore, the term $(S + dS)d\hat{\theta} + Gd\emptyset$ should normally be nonexistent or zero. However, since we are considering the case of a pension fund, then, without any loss of generality, we can argue that the self financing condition ensures that the additional term $(S + dS)d\hat{\theta} + Gd\emptyset$ comes from the contributions u made by the subscriber during the accumulation phase and it is used to finance the pension payments v during the decumulation period. Accordingly, therefore,

$$(S + dS)d\hat{\theta} + Gd\emptyset = mdt \tag{6a}$$

where m is the rate at which money enters and leaves the fund, and so

$$dW = (Wr + \phi(\mu_S - r) + m)dt + \phi\sigma_S dz_S \quad (6b)$$

where $\phi = \hat{\theta}S = \alpha W$ is the amount of wealth apportioned to the risky-asset class, $dS = S(\mu_S dt + \sigma_S dz_S)$ and $dG = Grdt$ represent the price process for the risky-asset class and the riskless-asset class respectively.

Now, since we want to perform the analysis of a pension fund in a regret theoretic framework, we will not only consider the outcome of the decision/choice made by the pension fund, but we shall also consider the outcome of another choice that the pension fund could have made. In particular, we consider a case in which the pension fund could have made a choice of taking no position in the risky-asset class, i.e. investing solely in the riskless-asset class. The wealth process or outcome of such a choice/decision would be

$$W^o(t) = \emptyset(t)G(t) \quad (7a)$$

and its associated stochastic differential equation would be

$$dW^o = \emptyset dG + Gd\emptyset \quad (7b)$$

Self-financing condition would then imply that

$$Gd\emptyset = mdt \quad (8a)$$

where m is as before, and so

$$dW^o = (W^o r + m)dt \quad (8b)$$

Equations (5a) –8(b) explicitly show the outcome of the actual investment choice made by the pension fund as well as the outcome of a forgone investment choice that the pension fund could have made. In what follows, we will now proceed and write down the objective function of the pension fund in a regret theoretic framework.

4. The Objective Function of the Pension Fund

The objective of the pension fund is to maximize the expected modified utility of its final wealth/final outcome at the death time τ of its subscriber, under the assumption that the pension fund anticipates to experience a feeling of regret if the outcome of its investment choice is less than the outcome of a forgone investment choice and that the pension fund takes this feeling into account when making its decision under uncertainty with the sole aim of curtailing future regrets. The forgone investment choice is taken as a full investment in the riskless-asset class. Following (4a), we can write the modified utility function of the pension fund as

$$\psi(W, W^o) = \Big(U(W) + f\big(U(W) - U(W^o)\big)\Big) = \Big(U\big(\hat{\theta}S + \emptyset G\big) + f\,\Big(U\big(\hat{\theta}S + \emptyset G - U\emptyset G \quad (9a)$$

Thus, the objective function of the pension fund is

$$\max_{\phi} \mathbb{E}_{t_0}^{\tau}\Big(\psi\big(W(\tau), W^o(\tau)\big)\Big) = \nabla \quad (9b)$$

where

$$\nabla = \max_{\phi} \mathbb{E}_{t_0}^{\tau}\left(U(\hat{\theta}S + \emptyset G) + f\left(U(\hat{\theta}S + \emptyset G) - U(\emptyset G) \right) \right)$$

i.e. finding the right allocation to the risky-asset class that will help maximize the expected modified utility of the pension fund's final wealth. Following [4] and [9], we assume the death time τ is independent of all other sources of risk. With this assumption, we can rewrite the above expected value as

$$\mathbb{E}_{t_0}^{\tau}\left(\psi(W(\tau), W^o(\tau)) \right) = \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} g(t)\psi(W(t), W^o(t))dt \right] \tag{10}$$

where $g(t) = n(t)p_{t_0}e^{-\rho(t-t_0)}$ is the actuarial discount factor, $n(t)$ is the instantaneous mortality rate (known as mortality force), p_{t_0} is the survival probability and ρ is the positive intertemporal discount rate (Battochio, Menoncin, & Scaillet, 2007).

Now, we need to define the utility function $U(*)$ and the regret function $f(*)$ that we shall use in the course of our analysis. The one most widely used utility function in the literature is the constant relative risk aversion utility function of the form $U(X) = \frac{X^{1-\vartheta}}{1-\vartheta}$ with $1-\vartheta < 1$. Here, we shall use a slight modification of this utility function. Now, we know that a pension function derives utility from its wealth after all contributions have been made and pensions have been paid. Therefore, if we let $M(t)$ represent all contributions and payments up to the present time, where $M(t)$ can be written as [4].

$$M(t) = \int_{t_0}^{t} m(s)e^{-r(s-t)}ds$$

then the pension fund will derive some utility from $W(t) - M(t)$, and so the utility function must be defined on this argument, i.e. $U(W(t) - M(t))$. This is our slight modification of the utility function. For the regret function $f(*)$, we assume that it is of the form $f(Y) = (Y + a)^{\rho}$, with $a > 0$ and $\rho > 0$, both of which are positive and less than 1 (Mhiri, & Prigent, 2010). Accordingly, therefore, the modified utility function can be written as

$$\psi(W, W^o) = \frac{(W(t)-M(t))^{1-\vartheta}}{1-\vartheta} + \left(\frac{1}{1-\vartheta}\left((W(t) - M(t))^{1-\vartheta} - (W^o(t) - \right.\right.$$
$$Mt1-\vartheta+a\rho \tag{11}$$

where all variables are as previously defined.

4.1. The Optimization Problem of the Pension Fund

The previous sections set the ground for formulating the optimization problem of the pension fund. As we assume that the pension fund seeks to maximize the expected modified utility of its terminal wealth after all contributions have been made and pensions have been made, we write the regret theoretic asset allocation problem as

$$\begin{cases} \max_{\phi} \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} g(t)\left(\frac{(W(t)-M(t))^{1-\vartheta}}{1-\vartheta} + \left(\frac{1}{1-\vartheta}\left((W(t) - M(t))^{1-\vartheta} - (W^o(t) - M(t))^{1-\vartheta} \right) + a \right)^{\rho} \right) dt \right] \\ dW = (Wr + \phi(\mu_S - r) + m)dt + \phi\sigma_S dz_S \\ dW^o = (W^o r + m)dt \\ W(t_0) = W_0 \\ W^o(t_0) = W_0^o \end{cases} \tag{12}$$

where $g(t)$ is the actuarial discount factor.

Following [4], let us define the function

$$\left\{V(t,W,W^0,\phi)=\left[\int_t^\infty g(s)\left(\frac{\left(W(s)-M(s)\right)^{1-\vartheta}}{1-\vartheta}+\left(\frac{1}{1-\vartheta}\left(\left(W(s)-M(s)\right)^{1-\vartheta}-\left(W^o(s)-M(s)\right)^{1-\vartheta}\right)+a\right)^\rho\right)ds\right]\right.$$

then the value function can be written as

$$J(t,W,W^0)=\max_{\phi} V(t,W,W^0,\phi)$$

This value function satisfies the HJB Hamilton-Jacobi-Bellman equation

$$J_t+\max_{\phi}\left[\left\{\begin{array}{c} g(t)\left(\frac{\left(W(t)-M(t)\right)^{1-\vartheta}}{1-\vartheta}+\left(\frac{1}{1-\vartheta}\left(\left(W(t)-M(t)\right)^{1-\vartheta}-\left(W^o(t)-M(t)\right)^{1-\vartheta}\right)+a\right)^\rho\right)\\ +\\ J_W(Wr+\phi(\mu_S-r)+m)+\frac{1}{2}J_{WW}\phi^2\sigma_S^2+J_{W^0}(W^or+m)\end{array}\right.\right]=0$$

From this equation, the first order condition for the maximization yields

$$\left[\left\{\begin{array}{c} g(t)\left(W(t)-M(t)\right)^{-\vartheta}\left(1+\rho\left(\frac{1}{1-\vartheta}\left(\left(W(t)-M(t)\right)^{1-\vartheta}-\left(W^o(t)-M(t)\right)^{1-\vartheta}\right)+a\right)^{\rho-1}\right)\\ +\\ J_W\mu_S+J_{WW}\sigma_S^2\phi\end{array}\right.\right]=0$$

$$\Rightarrow \phi^*=-\frac{\aleph}{J_{WW}\,\sigma_S^2} \tag{13}$$

where the subscripts indicate the partial derivatives with respect to J and

$$\aleph=\left\{\begin{array}{c} g(t)\left(W(t)-M(t)\right)^{-\vartheta}\left(1+\rho\left(\frac{1}{1-\vartheta}\left(\left(W(t)-M(t)\right)^{1-\vartheta}-\left(W^o(t)-M(t)\right)^{1-\vartheta}\right)+a\right)^{\rho-1}\right)\\ +\\ J_W\mu_S\end{array}\right.$$

The first order conditions are necessary and sufficient for optimality because the modified utility is concave in W and hence in ϕ, since ϕ is linked to W through $\phi=\hat{\theta}S=\alpha W$ by comparing equations 5b and 6b. In fact, under all the suitably stated conditions that must hold for the optimization problem of the pension fund, the modified utility function, and hence the value function, is increasing and concave in W. The proof is given in the last Appendix.

After substituting the expression of ϕ^*, we obtain the resulting Hamilton-Jacobi-Bellman equation,

$$\left[\left\{\begin{array}{c} J_t+g(t)\left(\frac{\left(W(t)-M(t)\right)^{1-\vartheta}}{1-\vartheta}+\left(\frac{1}{1-\vartheta}\left(\left(W(t)-M(t)\right)^{1-\vartheta}-\left(W^o(t)-M(t)\right)^{1-\vartheta}\right)+a\right)^\rho\right)\\ +\\ J_W(Wr+m)+\frac{\aleph}{J_{WW}\sigma_S^2}\left(\frac{1}{2}\aleph-J_W(\mu_S-r)\right)+J_{W^0}(W^or+m)\end{array}\right.\right]=0$$

For the value function, we can try the substitution $J(t,W,W^0)=h(t)g(t)\,\psi(W,W^o)$, where $h(t)$ is a function that needs to be determined [4]. From this substitution, we have

$$J_t=g(t)\psi(W,W^o)\,\frac{\partial h(t)}{\partial t}+\,h(t)\psi(W,W^o)\frac{\partial g(t)}{\partial t}+h(t)g(t)\,\frac{\partial\psi(W,W^o)}{\partial t}$$

$$J_W = h(t)g(t)\frac{\partial\psi(W,W^o)}{\partial W},\; J_{WW} = h(t)g(t)\frac{\partial}{\partial W}\frac{\partial\psi(W,W^o)}{\partial W}, J_{W^o} = h(t)g(t)\frac{\partial\psi(W,W^o)}{\partial W^o}$$

Plugging these expressions into the Hamilton Jacobi Bellman equation and simplifying terms, we obtain that $h(t)$ satisfies

$$\frac{\partial h(t)}{\partial t} + \left(\frac{1}{g(t)}\frac{\partial g(t)}{\partial t} + r\left(1 + \frac{1}{2\sigma_S^2}\frac{r}{(1-\vartheta)(\rho-1)}g(t)\right) + \frac{1}{2\sigma_S^2}\frac{g(t)}{h(t)}\frac{1}{(\rho-1)}\right)h(t) + A(W,W^o) = 0$$

Where

$$A(W,W^o) = \frac{1}{2}\frac{\aleph^2}{J_{WW}\sigma_S^2} - \frac{J_W}{J_{WW}}\frac{\aleph(\mu_S - r)}{\sigma_S^2}$$

The precise form of the function $h(t)$ is not important for computing the optimal portfolio composition of the pension fund since the Arrow-Pratt risk aversion index computed on $J(W,W^o)$ does not depend on $h(t)$. This makes it possible for us to obtain the composition of the optimal asset allocation of the pension fund. Thus, given a pair (u,v) of constant contribution and pension rates satisfying the feasibility condition, the composition of the optimal asset allocation of a pension fund which maximizes the expected modified utility of its final wealth is given by

$$\phi^* = \frac{E}{(L-H)\sigma_S^2} \tag{14}$$

Where

$$\begin{cases} E = \left(\frac{1}{g(t)} + \mu_S\right)\left[1 + \frac{\rho(\rho-1)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-1)\left(W^{o-\vartheta} - W^{-\vartheta}\right)M + a\rho(\rho-1)\right] \\ L = \vartheta\left[-1 + \frac{\rho(\rho-1)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-1)\left(W^{o-\vartheta} - W^{-\vartheta}\right)M + a\rho(\rho-1)\right] \\ H = \rho(\rho-1)\left(\frac{1}{W^\vartheta} + \frac{M\vartheta}{W^{\vartheta+1}}\right)\left[1 + \frac{\rho(\rho-2)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-2)\left(W^{o-\vartheta} - W^{-\vartheta}\right)M + a\rho(\rho-2)\right] \end{cases}$$

Without any loss of generality, and for simplicity, we can disregard the last term (set a to zero) in each of the above expressions, replace M with its value and then break each expression into two parts to get

$$\begin{cases} E_u = \left(\frac{1}{g(t)} + \mu_S\right)\left[1 + \frac{\rho(\rho-1)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-1)\left(W^{o-\vartheta} - W^{-\vartheta}\right)\int_0^t u\,\mathbb{I}_{t<T}e^{-r(s-t)}ds\right] \\ L_u = \vartheta\left[-1 + \frac{\rho(\rho-1)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-1)\left(W^{o-\vartheta} - W^{-\vartheta}\right)\int_0^t u\,\mathbb{I}_{t<T}e^{-r(s-t)}ds\right] \\ H_u = \rho(\rho-1)\left(\frac{1}{W^\vartheta} + \frac{\vartheta}{W^{\vartheta+1}}\int_0^t u\,\mathbb{I}_{t<T}e^{-r(s-t)}ds\right)\left[1 + \frac{\rho(\rho-2)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-2)\left(W^{o-\vartheta} - W^{-\vartheta}\right)\left(\int_0^t u\,\mathbb{I}_{t<T}e^{-r(s-t)}ds\right)\right] \end{cases}$$

$$\phi_u^* = \frac{E_u}{(L-H)\sigma_S^2} \tag{14a}$$

and

$$\begin{cases} E_v = \left(\frac{1}{g(t)} + \mu_S\right)\rho(1-\rho)\left(W^{o-\vartheta} - W^{-\vartheta}\right)\int_0^t v(1-\mathbb{I}_{t<T})e^{-r(s-t)}ds \\ L_v = \vartheta\rho(1-\rho)\left(W^{o-\vartheta} - W^{-\vartheta}\right)\int_0^t v(1-\mathbb{I}_{t<T})e^{-r(s-t)}ds \\ H_v = \rho^2(1-\rho)(\rho-2)\frac{\vartheta}{W^{\vartheta+1}}\left(\int_0^t v(1-\mathbb{I}_{t<T})e^{-r(s-t)}ds\right)^2\left(W^{o-\vartheta} - W^{-\vartheta}\right) \end{cases}$$

$$\phi_v^* = \frac{E_v}{(L-H)\sigma_S^2} \tag{14b}$$

so that

$$\phi^* = \phi_u^* + \phi_v^*$$

where u and v are linked by the feasibility condition and $L = L_u + L_v$ and $H = H_u + H_v$.

The first set of equations depends explicitly on the contribution rate u and the wealth level of the pension fund as a result of its actual investment choice, i.e. investing a positive amount in the risky- asset class. It also depends on the wealth level of a forgone investment choice, i.e. the investment choice that could have been made, which would involve taking no position in the risky-asset class. The wealth level of the forgone investment decision is a benchmark with which to compare the outcome or wealth level of the pension fund's actual investment choice. The second set of equations also depends on the two wealth levels as well as on the pension rate v. Furthermore, the mortality risk τ enters the maximization problem through the link that exists between the contribution rate u and the pension rate v in the feasible condition derived in Appendix 1. It is very pertinent to further stress that u and v must satisfy this feasibility condition as we have already demonstrated. As noted in (Heybati, Roodposhti, & Moosavi, 2011), if this link between u and v is completely not considered through the feasibility condition, then the composition of the optimal asset allocation of a pension fund is independent of mortality risk. Such an optimal asset allocation strategy for a pension fund can produce very shallow and extremely restricted results in practice.

A very important deduction from the two sets of equations is that the optimal allocation of the pension fund does explicitly depend on the wealth levels. This becomes evident since the equations themselves depend on the wealth levels and when we substitute the expressions of the equations into the composition of the optimal asset allocation of the pension fund, we get that it also depends on the wealth levels. From this, therefore, we can deduce that it is suboptimal for a pension fund to manage the accumulation phase, when contributions are made, and the decumulation phase, when pensions are paid, separately. Hence our model prohibits the pension fund from outsourcing any of the phases of the pension fund management to a second or third party. This means that the idea of outsourcing, commonly employed by large pension funds in emerging economies, is not effective and does not conform to an optimal strategy in a regret theoretic framework. Ideally, a firm should commit the entire management of its employees' retirement plan to the same pension fund and the pension fund itself must not outsource any of the phases of the pension management process. If it does, its action will be suboptimal and may have undesirable consequences. We now investigate what happens at the two phases involved in the management of a pension fund – accumulation and decumulation phases.

4.2. Accumulation Phase

When $t \leq T$, the pension fund is in the accumulation phase and the optimal allocation strategy in this phase can be written as

$$E_u = \left(\frac{1}{g(t)} + \mu_S\right)\left(1 + \frac{\rho(\rho-1)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-1)\left(W^{o\,-\vartheta} - W^{-\vartheta}\right)\frac{u}{r}\left(e^{rt}-1\right)\right)$$

$$L_u = \vartheta\left(-1 + \frac{\rho(\rho-1)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-1)\left(W^{o\,-\vartheta} - W^{-\vartheta}\right)\frac{u}{r}\left(e^{rt}-1\right)\right)$$

$$H_u = \rho(\rho-1)\left(\frac{1}{W^{\vartheta}} + \frac{u}{r}\frac{\vartheta}{W^{\vartheta+1}}\left(e^{rt}-1\right)\right)\left(1 + \frac{\rho(\rho-2)}{1-\vartheta}\left(W^{1-\vartheta} - W^{o\,1-\vartheta}\right) + \rho(\rho-2)\left(W^{o\,-\vartheta} - W^{-\vartheta}\right)\frac{u}{r}\left(e^{rt}-1\right)\right)$$

$$\phi_u^* = \frac{E_u}{(L-H)\sigma_S^2} \text{ and } \phi_v^* = 0 \text{ since } t \leq T \Rightarrow \left(\int_0^t v\left(1-\mathbb{I}_{t<T}\right)e^{-r(s-t)}ds\right) = 0$$

We first recall that μ_S can be positive, negative or zero and $\left(\frac{1}{g(t)} + \mu_S\right)>0$ since g is a discount factor whose value is positive and less than 1. Suppose now that the pension fund is not regret averse. During the accumulation phase, and in the absence of regret aversion, i.e. ρ=0, the optimal allocation to the risky-asset class assumes negative values and thus contains a decreasing proportion of the risky-asset class with respect to time. This means that, in the absence of regret aversion, the optimal allocation to risky assets during the accumulation phase decreases through time in order for the pension fund to meet payments of future pensions to its subscribers. This property is not difficult to check. For instance, when we set the regret coefficient to zero and differentiate the resulting expression with respect to time, we obtain

$$\frac{d\phi_u^*}{dt} = \frac{1}{g^2}\frac{dg}{dt} < 0 \;\forall\; \frac{dg}{dt} < 0$$

as previously noted. The graphical illustration is shown as below

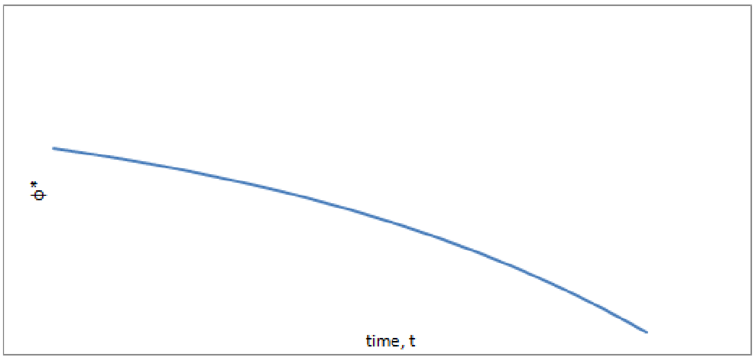

Figure 4. *Behavior of the optimal asset allocation with time*

This behavior during the accumulation phase substantiates the implications of the results documented in Battocio, Menoncin & Scaillet (2003; 2007). However, in the presence of regret, as in our case, the conclusion is not always the same. Indeed, we notice that, depending on the level of aversion and the choice of the underlying variables, the optimal asset allocation to the risky-asset class takes positive values and thus contains a non-decreasing proportion of the risky-asset class with regard to time. This means that, in the presence of regret aversion, regret averse pension funds have an optimal risky-asset allocation strategy that increases through time during the accumulation phase. In our view, the intuition behind this is pragmatically clear. Regret averse pension funds invest more in risky assets as time passes so as not to miss the upside potential of the risky-asset market, especially in bullish times. Thus a regret averse pension fund invests an increasing amount of wealth in the risky asset class. This is done in order to have a higher return on the managed wealth and on the contributions made by the subscribers.

4.3. Decumulation Phase

When $t > T$, the pension fund is in the decumulation phase and the optimal asset allocation strategy in this phase can be written as

$$E_v = \frac{v}{r}\left(\frac{1}{g(t)} + \mu_S\right)\rho(1-\rho)\left(W^{o-\vartheta} - W^{-\vartheta}\right)\left(e^{r(t-T)} - 1\right)$$

$$L_v = \frac{v}{r}\vartheta\rho(1-\rho)\left(W^{o-\vartheta} - W^{-\vartheta}\right)\left(e^{r(t-T)} - 1\right)$$

$$H_v = \left(\frac{v}{r}\right)^2 \rho(1-\rho)\frac{\vartheta}{W^{\vartheta+1}}\left(e^{r(t-T)} - 1\right)\left(\rho(\rho-2)\left(W^{o-\vartheta} - W^{-\vartheta}\right)\left(e^{r(t-T)} - 1\right)\right)$$

$$\Phi_v^* = \frac{\left(\frac{1}{g(t)} + \mu_S\right)}{\left(\vartheta - \frac{v}{r}\rho(\rho-2)\frac{\vartheta}{W^{\vartheta+1}}\left(e^{r(t-T)} - 1\right)\right)\sigma_S^2}$$

During the decumulation phase, and in the absence of regret aversion, i.e. ρ=0, the optimal allocation to the risky-asset class takes positive values and therefore increases through time when the actuarial discount factor decreases with time. This property is very easy to check. Indeed, if we set the coefficient of regret aversion to zero, we obtain

$$\Phi_v^* = \frac{\left(\frac{1}{g(t)} + \mu_S\right)}{\vartheta} > 0$$

which shows that the risky-asset class takes positive values. Furthermore, if we take the time derivative of this expression, we see that

$$\frac{d\Phi_v^*}{dt} = \frac{-1}{g^2}\frac{dg}{dt} > 0 \ \forall \ \frac{dg}{dt} < 0$$

which shows that the optimal allocation to the risky-asset class increases through time. Again, this conforms to the deductions made in Battocio, Menoncin & Scaillet (2003; 2007). This behavior is depicted below.

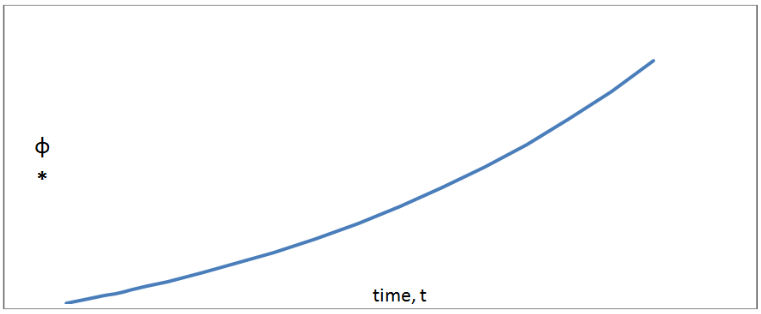

Figure 5. *Behavior of the optimal asset allocation with time*

In the presence of regret, as in our case, the situation is somewhat similar. Indeed, for all admissible levels of regret aversion, the optimal allocation to the risky-asset class takes only positive values during the decumulation phase and increases through time. The optimal allocation is skewed in favor of the risky-asset class in the decumulation phase. In fact when the retirement date T is still far away, the pension fund can afford to invest in the risky-asset class because of the belief that the risky-asset class may provide a better opportunity for it to accumulate more wealth before the retirement date. This behavior is borne out of regret aversion. The pension fund is averse to regret because it does not want to experience the feeling of regret that comes when the risky-asset appreciates over time and the pension fund does not have a high position in it. We recall from the formulation of our problem that the pension fund feels regret or rejoicing with respect to the risky-asset class. Therefore, if the pension fund anticipates an ex-post feeling of regret and factors this feeling into its decision making process, then to maximize its portfolio value, increase its final wealth level and minimize its future regret the pension fund's optimal strategy would be to take an increasing position in the risky-asst class during both the accumulation and decumulation phases.

We must emphasize that this result contrasts the behavior of a pension fund in the traditional expected utility (EU) framework. In the EU framework, the optimal allocation to risky assets decreases through time in the accumulation phase so that the pension fund would be able to make a sure and substantial pension payment in the decumulation phase, while the optimal asset allocation to risky assets increases through time in the decumulation phase. In our case, however, the allocation to the risky-asset class soars with time during both phases. In particular, this is so in the decumulation phase because after retirement and during the payment of pensions, the higher the number of pension installments paid, the fewer remaining pensions left to be paid, and as the death time approaches, the probability that the pension subscriber would die increases, so the pension fund can accept to take much risk with less feeling of regret ex-post. The pension fund can afford to invest more and more in the risky-asset class in order to have a high return on the received contributions from the subscribers (which are then reinvested for the purpose of growing the contributed funds) and on the total managed wealth. This behavior is borne out of regret. Regret aversion forces the pension fund to behave this way so as to minimize the feeling of regret that would occur if the risky-asset goes up and the pension fund is not there.

When the pension fund starts paying pensions, the higher the number of pension installments paid, the lower the probability to pay another pension installment since the death probability increases through time. Thus, after the retirement date T when t increases, the pension fund can afford to invest increasingly in the risky-asset class because it has fewer pensions to pay and thus would deem it optimal to increase the allocation to risky assets because of the upside potential they are capable of generating. Thus, for a pension fund which seeks to maximize the expected modified utility of its final wealth, the optimal asset allocation rule is such that the amount allocated to risky-asset class increases through time during both phases- the accumulation phase and the decumulation phase.

5. Conclusion

Battocchio *et al.,* (2007) use the theory of expected utility-maximization, applied to the management of two phases of a pension fund -the accumulation and decumulation phases – to conclude that the optimal asset allocation to risky assets in both phases must be different- it must decrease through time in the accumulation phase and increase through time in the decumulation phase. Instead, in this paper, we have considered the problem of finding the optimal asset allocation of a pension fund in a regret theoretic framework, where our intuition is motivated by studies that find support for the feelings of regret in asset allocation and investment decision making. Accordingly, we incorporate regret into the decision-making process of a pension fund and derive the optimal asset allocation of a pension fund in a regret theoretic framework. We focus on the composition of the optimal asset allocation strategy in the accumulation and decumulation phases of the pension fund management. The configuration of the financial market is such that there is a risky-asset class whose price follows a geometric Brownian motion, a riskless-asset class paying a non trivial interest rate and we assume the market is not necessarily complete. Furthermore, it is assumed the pension fund possesses a power utility function and a regret/rejoice function which satisfies all the usual properties of a standard regret function. With an emphasis on regret, we derive approximated closed-form solutions for the pension fund and analyze the optimal asset allocation in the accumulation and decumulation phases. We consider a random death time of the representative subscriber which we assume to follow a Log-logistic distribution. Under this assumption, we derive the feasibility condition connecting the constant contribution rate u, constant pension rate v, and the random death time τ. This is a major area of contribution in this paper. There are three risk attributes in our objective function. The first is the traditional volatility, the second is the regret risk, embedded in the regret function, and the third is the mortality risk that comes in through the feasibility condition.

We show that the optimal asset allocation strategy for a representative regret averse pension fund in the accumulation phase is not different from the strategy in the decumulation phase. This is another major area of contribution in this paper. This particular result contrasts sharply with that obtained in the expected utility framework in which the optimal asset allocation in the accumulation phase is completely different from the optimal asset allocation in the decumulation phase. In particular, under the expected utility framework, the optimal allocation to risky assets decreases with time in the accumulation phase so as to make it possible for the pension fund to guarantee the payment of pensions to the representative subscriber after retirement while, in the decumulation phase when pensions are being paid, the optimal allocation to risky assets increases with time. Instead, in this paper, and under the regret theory framework, we find that the optimal allocation to risky-asset class increases during both accumulation and decumulation phases. The intuition to support this behavior is that when pensions are paid, the probability of paying more pensions decreases as time passes because of the subscriber's inevitable closeness to death after the retirement date. This makes it possible for the pension fund to invest the remaining available wealth in more and more risky assets and so the allocation to risky assets increases with time. The pension fund is confident to take higher positions in risky assets because of its reducing level of obligations and desire to benefit from the upside potential of the risky-asset market. We observe that the optimal asset allocation of the pension fund jointly depends on its global wealth levels in both accumulation and decumulation phases. Therefore, it is suboptimal for a pension fund to manage the accumulation and decumulation phases separately, a result which is standard in the literature. Outsourcing the management of just a phase of a pension fund is therefore not an optimal strategy. A pension fund that manages the accumulation phase should ideally manage the decumulation phase.

Future research would be an extension of our model to cater to the optimal asset allocation rule of a pension fund that manages the retirement proceeds of immortal subscribers in a regret theoretic framework. Although humans are not immortal and such extensions would be largely theoretical, with little practical implications and distorted real-life consequences, yet we believe the outcome would be inspiring to theoretical analysts and researchers who are interested in immortality bias in survival analysis as regards the management of pension funds, especially given that many countries globally have begun to significantly increase employee retirement age because medical breakthrough has, over the last decades, led to unprecedented improvements in life expectancy.

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Appendices

Appendix 1: *Derivation of the Feasibility Condition*

If u represents the constant contribution rate to a pension fund and v represents the constant pension rate paid to a subscriber, then the feasibility condition holds for the pair (u, v), $u > 0$ and $v > 0$ if

$\mathbb{E}\left[\int_0^\tau m(t)e^{-rt}dt\right] = 0$, and the resulting expression for the feasibility condition is

$$\frac{u}{v} = -1 + \frac{1 - \mathbb{E}[e^{-r\tau}]}{1 - \mathbb{E}[e^{-r\tau}\mathbb{I}_{\tau<T}] - e^{rT}\mathbb{P}(\tau \geq T)}$$

where τ is the random death time.

Proof

Given that T is the retirement date:

* When $t < T$, we have $\mathbb{I}_{t<T} = 1$ and $m(t) = \frac{dU(t)}{dt} = u > 0 \Rightarrow$ contributions are made, money enters the fund and so we are in the accumulation phase.

* When $t \geq T$, we have $\mathbb{I}_{t<T} = 0$ and $m(t) = -\frac{dV(t)}{dt} = -v < 0 \Rightarrow$ pensions are paid, money leaves the fund and so we are in the decumulation phase.

If we denote the expected present value of all pensions by $\mathbb{E}PVP$, where

$$\mathbb{E}PVP = \mathbb{E}\int_0^\tau v(1 - \mathbb{I}_{t<T})\, e^{-rt}dt$$

and the expected present value of all contributions by $\mathbb{E}PVC$, where

$$\mathbb{E}PVC = \mathbb{E}\int_0^\tau u\mathbb{I}_{t<T}\, e^{-rt}dt$$

then from the point of view of the subscriber, $\mathbb{E}PVP \geq \mathbb{E}PVC$, i.e.

$$A: \mathbb{E}\int_0^\tau v(1 - \mathbb{I}_{t<T})\, e^{-rt}dt \geq \mathbb{E}\int_0^\tau u\mathbb{I}_{t<T}\, e^{-rt}dt$$

and from the point of view of the pension fund, $\mathbb{E}PVP \leq \mathbb{E}PVC$, i.e

$$B: \mathbb{E}\int_0^\tau v(1 - \mathbb{I}_{t<T})\, e^{-rt}dt \leq \mathbb{E}\int_0^\tau u\mathbb{I}_{t<T}\, e^{-rt}dt$$

Accordingly, therefore, both parties reach an agreement when $A \cap B \neq \emptyset$, i.e.

$$\mathbb{E}\int_0^\tau v(1 - \mathbb{I}_{t<T})\, e^{-rt}dt \geq \mathbb{E}\int_0^\tau u\mathbb{I}_{t<T}\, e^{-rt}dt \;\cap\; \mathbb{E}\int_0^\tau v(1 - \mathbb{I}_{t<T})\, e^{-rt}dt \leq \mathbb{E}\int_0^\tau u\mathbb{I}_{t<T}\, e^{-rt}dt \neq \emptyset$$

$$\Rightarrow \mathbb{E}\int_0^\tau u\mathbb{I}_{t<T}\, e^{-rt}dt = \mathbb{E}\int_0^\tau v(1 - \mathbb{I}_{t<T})\, e^{-rt}dt$$

$$\Rightarrow \mathbb{E}\int_0^\tau \left(u\mathbb{I}_{t<T} - v(1 - \mathbb{I}_{t<T})\right) e^{-rt}dt = 0, \text{i.e.}$$

$$\mathbb{E}\left[\int_0^\tau m(t)e^{-rt}dt\right] = 0,$$

where $m(t) = u\mathbb{I}_{t<T} - v(1 - \mathbb{I}_{t<T})$, e^{-rt} is the discount factor and τ is the random death time of the subscriber. This is the feasibility condition that has to be satisfied before the subscriber and the pension fund can accept the pair (u, v). It guarantees that neither the pension fund nor the subscriber feels cheated.

We next prove the feasibility condition

$$\frac{u}{v} = -1 + \frac{1 - \mathbb{E}[e^{-r\tau}]}{1 - \mathbb{E}[e^{-r\tau}\mathbb{I}_{\tau<T}] - e^{rT}\mathbb{P}(\tau \geq T)}$$

Eliminating $m(t)$ from $\mathbb{E}\left[\int_0^\tau m(t)e^{-rt}dt\right] = 0$ and $m(t) = u\mathbb{I}_{t<T} - v(1 - \mathbb{I}_{t<T})$ yields

$$\frac{u}{v} = \frac{\mathbb{E}\left[\int_0^\tau e^{-rt}dt\right] - \mathbb{E}\left[\int_0^\tau \mathbb{I}_{t<T}e^{-rt}dt\right]}{\mathbb{E}\left[\int_0^\tau \mathbb{I}_{t<T}e^{-rt}dt\right]}$$

$$\frac{u}{v} = \frac{\mathbb{E}\left[\int_0^\tau e^{-rt}dt\right]}{\mathbb{E}\left[\int_0^\tau \mathbb{I}_{t<T}e^{-rt}dt\right]} - 1$$

Now, since we can write

$$\int_0^{\tau} e^{-rt}dt = \frac{1}{r} - \frac{e^{-r\tau}}{r}$$

and

$$\int_0^{\tau} \mathbb{I}_{t<T} e^{-rt}dt = \begin{cases} \dfrac{1-e^{-r\tau}}{r} & \text{for } \tau < T \\ \dfrac{1-e^{-rT}}{r} & \text{for } \tau \geq T \end{cases}$$

then we have

$$\mathbb{E}\left[\int_0^{\tau} e^{-rt}dt\right] = \frac{1}{r} - \frac{\mathbb{E}(e^{-r\tau})}{r} = \frac{1}{r}\left(1-\mathbb{E}(e^{-r\tau})\right)$$

and

$$\mathbb{E}\left[\int_0^{\tau} \mathbb{I}_{t<T} e^{-rt}dt\right] = \int_0^{T} \frac{1-e^{-r\tau}}{r} f(\tau) + \int_T^{\infty} \frac{1-e^{-rT}}{r} f(\tau)$$

where $f(\tau)$ is the density of the random death time τ.
Complete integration by parts yields

$$\mathbb{E}\left[\int_0^{\tau} \mathbb{I}_{t<T} e^{-rt}dt\right] = \frac{1}{r}\left(\mathbb{P}(\tau<T)+\mathbb{P}(\tau \geq T)\right) - \frac{1}{r}\int_0^{T} e^{-r\tau} f(\tau)d\tau - \frac{1}{r}e^{-rT}\mathbb{P}(\tau \geq T)$$

or

$$\mathbb{E}\left[\int_0^{\tau} \mathbb{I}_{t<T} e^{-rt}dt\right] = \frac{1}{r}\left(1-\mathbb{E}(e^{-r\tau}\mathbb{I}_{\tau<T}) - e^{-rT}\mathbb{P}(\tau \geq T)\right)$$

Thus

$$\frac{u}{v} = -1 + \frac{\left(1-\mathbb{E}(e^{-r\tau})\right)}{1-\mathbb{E}(e^{-r\tau}\mathbb{I}_{\tau<T}) - e^{-rT}\mathbb{P}(\tau \geq T)}$$

as required to prove.

Remarks

The proposition shows that pensions are proportional to contributions. This is exactly what is observed in practice; the higher the contributions towards retirement, the higher the pensions at retirement.

Appendix 2: Computation of the Feasibility Condition

Battocchio, Menoncin and Scaillet (2003, 2007) assume death time τ follows a Gompertz-Makeham distribution and a Weibull distribution and they compute the feasibility condition based on these assumptions. Here, we explicitly compute the feasibility condition by supposing that the death time τ follows a log-logistic distribution. Besides giving a concrete view of the feasibility condition, the log-logistic distribution takes only positive arguments, provides a good characterization of the death time and is most widely used in death/survival analysis. These are some of its desirable properties which explain why we favor it in our work.

The log-logistic distribution density function of the death time τ is given by

$$f(\tau) = \frac{\frac{\beta}{\alpha}\left(\frac{\tau}{\alpha}\right)^{\beta-1}}{\left[1+\left(\frac{\tau}{\alpha}\right)^{\beta}\right]^2} \text{ where } \tau > 0, \alpha > 0, \beta > 0$$

α and β are the scaling and shaping factor respectively.

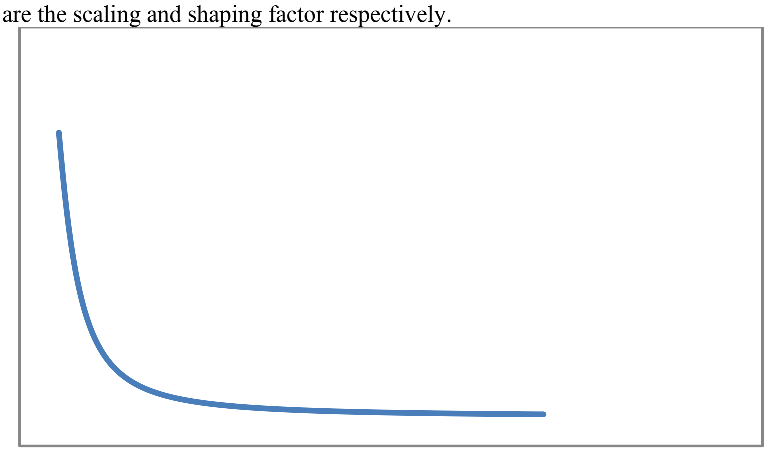


Figure 1: Log-logistic distribution of death time τ with $\alpha = 1.7,\ \beta = 1.3$

The expected time of death is given by

$$\int_0^{\infty} \tau f(\tau)d\tau = \alpha \mathrm{B}\left(1-\frac{1}{\beta}, 1+\frac{1}{\beta}\right) = \alpha\Gamma\left(1-\frac{1}{\beta}\right)\Gamma\left(1+\frac{1}{\beta}\right), \qquad \beta > 1$$

where B and Γ are the Beta and Gamma functions respectively. The behavior of the expected death time is shown in Figure 2 where we have set $\alpha \in [2, 10]$ and $\beta \in [1.10, 1.50]$. We see that the expected time of death soars to roughly 100 years when the parameters take the set values above.

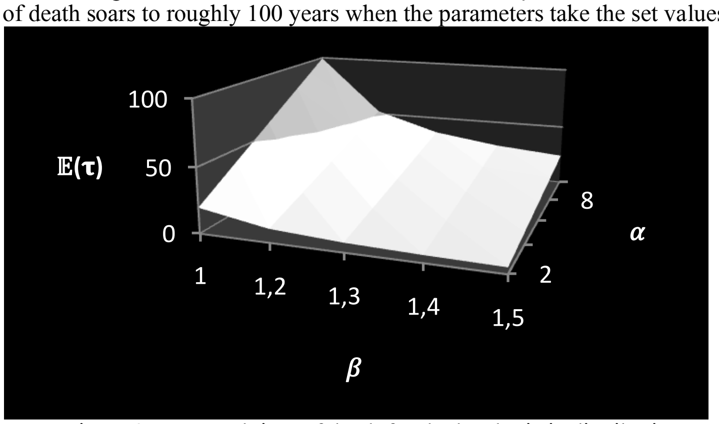


Figure 2: Expected time of death for the log-logistic distribution

Before we compute the feasibility condition, we need to first obtain $\mathbb{P}(\tau \geq T)$, $\mathbb{E}(e^{-r\tau})$ and $\mathbb{E}(e^{-r\tau}\mathbb{I}_{\tau<T})$ under our assumption of a log-logistic death time τ distribution.

$\mathbb{P}(\tau \geq T)$

The probability that the death time τ would occur on or after the obligatory retirement date T is given by

$$\mathbb{P}(\tau \geq T) = \frac{\beta}{\alpha}\int_T^{\infty} \frac{\left(\frac{\tau}{\alpha}\right)^{\beta-1}}{\left[1+\left(\frac{\tau}{\alpha}\right)^{\beta}\right]^2} d\tau$$

If we apply the change of variable $u = 1+\left(\frac{\tau}{\alpha}\right)^{\beta}$, where $u \to 0 \; as \; \tau \to \infty$, we will obtain

$$\mathbb{P}(\tau \geq T) = \frac{1}{1+\left(\frac{T}{\alpha}\right)^{\beta}}$$

We remark that if the retirement date is very far into the future, the probability of death time occurring after the retirement date tends to zero. This means that the death time is most likely to occur before the retirement date and thus there is a risk that the employee may die in active service before retirement. This explains why more countries are increasingly favoring earlier retirement dates.

$\mathbb{E}(e^{-r\tau})$

This is the expected value of the discounting factor over the death time of the subscriber. An advantage of the log-logic distribution is that it takes only positive death time, unlike a Normal distribution which can assume an undesirable negative death time. Considering this, we have

$$\mathbb{E}(e^{-r\tau}) = \frac{\beta}{\alpha}\int_0^{\infty} e^{-r\tau}\frac{\left(\frac{\tau}{\alpha}\right)^{\beta-1}}{\left[1+\left(\frac{\tau}{\alpha}\right)^{\beta}\right]^2} d\tau = \int_1^{\infty} u^{-2} e^{-r\alpha(u-1)^{\frac{1}{\beta}}} du$$

where we have used the change of variable $\tau = \alpha(u-1)^{\frac{1}{\beta}}$. Since this integral does not admit an elementary algebraic solution, we may propose an approximation as in the Proposition below.

Proposition 1

Under the assumption that $(u-1)^{\frac{1}{\beta}} < 1$ for $\beta > 0$, $\mathbb{E}(e^{-r\tau})$ approximates to

$$\mathbb{E}(e^{-r\tau}) \cong \frac{\beta}{r\alpha}\left[\left(\frac{1}{r\alpha}\right)^{\beta-1}\Gamma(\beta) - 2\left(\frac{1}{r\alpha}\right)^{2\beta-1}\Gamma(2\beta)\right]$$

where Γ is the complete gamma function defined as

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt = (s-1)!$$

Proof

Set $t = r\alpha(u-1)^{\frac{1}{\beta}}$ and use negative binomial theorem to expand $\left[1+\left(\frac{t}{r\alpha}\right)^{\beta}\right]^{-2}$ after which the result follows.

* Computing $\mathbb{E}(e^{-r\tau}\mathbb{I}_{\tau<T})$ is akin to computing $\mathbb{E}(e^{-r\tau})$ but with an added restriction that the death time must occur before the retirement date, i.e. given that the subscriber dies before the retirement date. Arguments similar to Proposition 2 give

$$\mathbb{E}(e^{-r\tau}\mathbb{I}_{\tau<T}) \cong \frac{\beta}{r\alpha}\left[\left(\frac{1}{r\alpha}\right)^{\beta-1}\int_0^T t^{\beta-1}e^{-t}dt - 2\left(\frac{1}{r\alpha}\right)^{2\beta-1}\int_0^T t^{2\beta-1}e^{-t}dt\right]$$

If we define the lower incomplete gamma function as

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$$

then we will have

$$\mathbb{E}(e^{-r\tau}\mathbb{I}_{\tau<T}) \cong \frac{\beta}{r\alpha}\left[\left(\frac{1}{r\alpha}\right)^{\beta-1}\gamma(\beta,T) - 2\left(\frac{1}{r\alpha}\right)^{2\beta-1}\gamma(2\beta,T)\right]$$

Accordingly, we plug these closed-form approximations for $\mathbb{P}(\tau \geq T)$, $\mathbb{E}(e^{-r\tau})$ and $\mathbb{E}(e^{-r\tau}\mathbb{I}_{\tau<T})$ into the feasibility condition to get

$$\frac{u}{v} \cong \frac{\beta\left[\left(\frac{1}{r\alpha}\right)^{\beta-1}\Gamma(\beta,T) - 2\left(\frac{1}{r\alpha}\right)^{2\beta-1}\Gamma(2\beta,T)\right] + r\alpha\frac{e^{-rT}}{\left[1+\left(\frac{T}{\alpha}\right)^{\beta}\right]}}{r\alpha - \beta\left[\left(\frac{1}{r\alpha}\right)^{\beta-1}\gamma(\beta,T) - 2\left(\frac{1}{r\alpha}\right)^{2\beta-1}\gamma(2\beta,T)\right] - r\alpha\frac{e^{-rT}}{\left[1+\left(\frac{T}{\alpha}\right)^{\beta}\right]}}$$

where $\Gamma(s,x)$ is the upper incomplete gamma function defined as

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt$$

and

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt = (s-1)!$$

This is the condition that has to be satisfied for the pension fund and the subscriber to agree on a pension contract when the death time of the subscriber is assumed to follow a log-logistic distribution. We present the results of the feasibility condition for several values of α, β, T and r in Table 1 below.

Table 1: Approximation for the feasible ratio

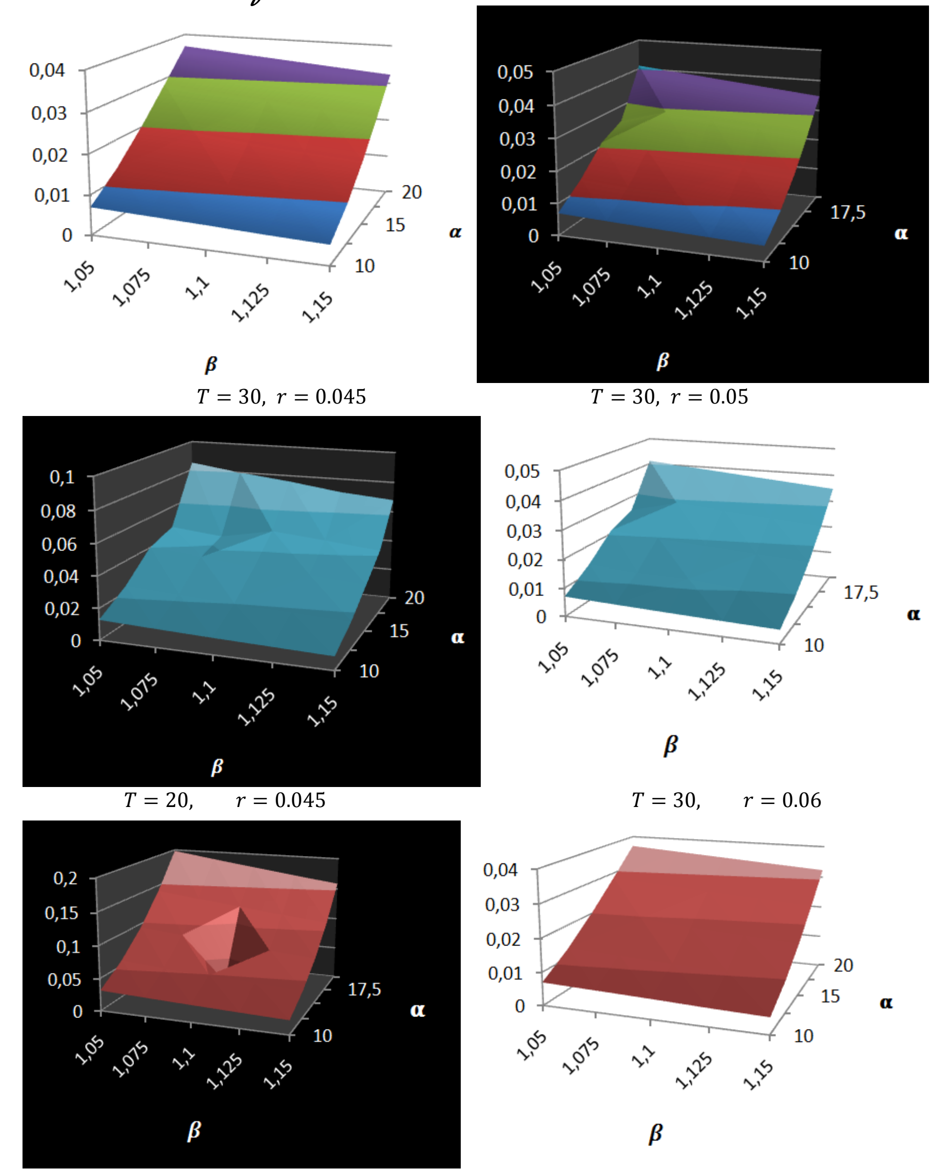
r	α	β	T	$\frac{u}{v}$
0.05	20	1.1	30	0.0074
0.05	20	1.1	20	0.0936
0.05	20	1.1	10	0.1322
0.05	20	1.05	30	0.2201
0.05	20	1,15	30	0.0651
0.05	20	1.20	30	0.0602
0.05	10	1.1	30	0.0095
0.05	15	1.1	30	0.0311
0.05	25	1.1	30	0.0721
0.04	20	1.1	30	0.0458
0.05	20	1.1	30	0.0987
0.06	20	1.1	30	0.1245

From Table 1 it is clear that, given the age of a subscriber, when the retirement date T increases, the feasible ratio $\frac{u}{v}$ decreases and so the pension fund can afford to pay a higher pension rate to the subscriber. In fact, the pension fund can demand lower contribution rates when the contributions are made for a long period of time. Furthermore, when the retirement date T is sufficiently far away, the feasible ratio $\frac{u}{v}$ is decreasing with respect to β and increasing with respect to α. Our result contrasts with Battocchio, Menoncin and Scaillet (2003, 2007) who find a decreasing relationship between $\frac{u}{v}$ and both α and β for a sufficiently large retirement date T.

Table 1 also shows that the higher the short term interest rate r the higher the feasible ratio and therefore the lower the pension rate the pension fund can afford to pay. In fact, when the interest rate increases it becomes more difficult for the pension fund to meet future payments. This will consequently force the pension fund to demand higher contribution rates. Again our result contrasts sharply with Battocchio, Menoncin and Scaillet (2003, 2007) who find an inverse relationship between the feasible ratio $\frac{u}{v}$ and the short term interest rate. We also find that, as the retirement date T increases, the probability that the death time will occur after the retirement date decreases. This means that more and more, it gets more and more likely that the subscriber will die before retirement or while in service.

Using the derived approximated ratio, we can graphically depict the behavior of the ratio $\frac{u}{v}$ with respect to the parameters α and β. These graphs are shown in Figure 3, where three different values of T and r are chosen. In particular, $T = 10, 20, 30$ and $r = 0.05, 0.06, 0.07$ and the values of α and β belong to $[10, 20]$ and $[1.05, 1.15]$ respectively.

Figure 3: Feasible ratio $\frac{u}{v}$



$T = 30,\ r = 0.045$ $T = 30,\ r = 0.05$

$T = 20,\ r = 0.045$ $T = 30,\ r = 0.06$

$T = 10,\ r = 0.045$ $T = 30,\ r = 0.07$

The first column of Figure 3 shows the behavior of the feasible ratio $\frac{u}{v}$ for $T \in \{10, 20, 30\}$, while the second column analyzes how $\frac{u}{v}$ changes for $r \in \{0.05, 0.06, 0.07\}$. We notice from the second column of Figure 3 that changes in r does not markedly affect the shape of $\frac{u}{v}$. As such, the interest rate r only affects the magnitude or level of $\frac{u}{v}$ without concomitantly altering its behavior with respect to other parameters.

Appendix 3. (a) The Modified Utility Function

Contrary to other decision criteria such as the minimax and maximin criteria, regret theory takes account of occurrence probabilities of different events and this aids the modification of utility functions. Loomes and Sugden (1982) propose a regret/rejoice function for pairs of lotteries involving different events and their occurrence probabilities.

Let $L_x = [(x_1, p_1), \dots, (x_n, p_n)]$ and $L_y = [(y_1, p_1), \dots, (y_n, p_n)]$ be two lotteries such that L_x is chosen. Let x and y, with utility functions $U(x)$ and $U(y)$, be the outcomes generated by L_x and L_y respectively. The difference between the utility functions of these two outcomes quantifies regret or rejoicing. Thus, if we define this difference by $\vartheta(x, y)$, then

$$\vartheta(x, y) = U(x) - U(y)$$

If $x < y$ so that $U(x) < U(y)$, then $U(x) - U(y) < 0$ and $\vartheta(x, y)$ is negative, which implies that an agent experiences a feeling of regret because the outcome y of a forgone choice dominates the outcome x of his choice. Conversely, if $x > y$ so that $U(x) > U(y)$, then $U(x) - U(y) > 0$ and $\vartheta(x, y)$ is positive, which implies that the agent rejoices because the outcome x of his choice dominates the outcome y of an alternative choice. Thus, regret theory assumes that not only does an agent experience regret, but also that the anticipation of experiencing regret is factored into the decision making process. If the agent is indifferent between both choices, then he is indifferent between their outcomes and so $x \sim y$ or $(x) = U(y)$ or $\vartheta(x, y) = 0$, which implies that the agent experiences no regret or rejoicing. Essentially, therefore, we assume that the agent is not indifferent between outcomes.

The modified utility function of an outcome x resulting from a given choice L_x in a particular state of the world s in the presence of another outcome y resulting from an alternative choice L_y in the same state of the world, is given by

$$\psi(x, y) = U(x) + f\big(U(x) - U(y)\big)$$

where $\psi(x, y)$ is the modified utility of achieving x, knowing that y could have been achieved by making a different choice in the same state of the world, $U(x)$ is the traditional choiceless utility function that an agent would derive from outcome x if he experienced it without having to choose or make a choice. That is, the utility derived from the outcome of a choice without considering the outcome of a different choice that could have been made in the same state of the world. It is essentially the utility function in the traditional expected utility theory that concerns only the outcome of an agent's choice. Furthermore, $U(x)$ is increasingly monotone and concave, i.e. $U'(x) > 0$ and $U''(x) < 0$, showing that agents prefer more to less and exhibit risk aversion. The difference $U(x) - U(y)$ is the utility loss or gain of having outcome x rather than a forgone outcome y, while $f\big(U(x) - U(y)\big)$ specifies the regret of having outcome x rather than a forgone outcome y. The regret function $f\big(\vartheta(x, y)\big)$ is monotonically increasing, so that $f\big(\vartheta(x, y)\big) > f(0)$ whenever $\vartheta(x, y) > 0$ and $f\big(\vartheta(x, y)\big) < f(0)$ whenever $\vartheta(x, y) < 0$; it is decreasingly concave and thrice differentiable, so that $f''\big(\vartheta(x, y)\big) < 0$ (regret aversion) whenever $\vartheta(x, y) < 0$ and $f'''\big(\vartheta(x, y)\big) > 0$; it assumes a value of zero when an agent is indifferent between outcomes x and y, so that $f\big(\vartheta(x, y)\big) = 0$ whenever $\vartheta(x, y) = 0$. Regret theory assumes that the degree of regret or rejoicing that an agent experiences depends only on the difference between the choiceless utility of 'what is' and the choiceless utility of 'what might have been if another choice had been made or if another course of action had been taken'. This is why we have f defined on $\vartheta(x, y)$. The regret-rejoice function $f\big(\vartheta(x, y)\big)$ assigns a real-valued number to every possible increase or decrease of choiceless utility. Notice that, an agent who does not feel regret or rejoicing, i.e. when $f\big(\vartheta(x, y)\big)$ is zero, constant or linear, will simply maximize his expected choiceless utility. This special case of regret theory corresponds to the traditional expected utility theory. To assume that agents maximize expected modified utility is to generalize the traditional expected utility theory in an intuitively natural way, since the agent who does experience regret and rejoicing can be expected to try to anticipate those feelings and take them into account in his objective function when making a decision under uncertainty.

We now provide some intuitive explanations behind these assumptions.

$$= 0 \text{ whenever } \vartheta(x, y) = 0$$

We know, from the preceding argument, that $\vartheta(x, y) = 0 \Leftrightarrow U(x) = U(y) \Leftrightarrow x \sim y$. Therefore, the assumption implies that an agent experiences no regret when he is indifferent between two choices and their outcomes. So, $f(0) = 0$.

$$f''\big(\vartheta(x, y)\big) < 0 \text{ whenever } \vartheta(x, y) < 0$$

We know that $\vartheta(x, y) < 0 \Leftrightarrow U(x) < U(y) \Leftrightarrow x < y$, meaning that the outcome x of the agent's choice is dominated by the outcome y of a forgone alternative. This causes the agent to experience a feeling of regret and thus develop an aversion towards regret. It is this aversion that results in the concavity of the regret function. Therefore, $f'' < 0$ whenever $\vartheta(x, y) < 0$ and $f'' > 0$ whenever $\vartheta(x, y) > 0$.

is monotonically increasing

This comes from the fact that an agent rejoices more when his outcome turns out more favorable than an alternative outcome and regrets more when his outcome turns out less favorable than an alternative outcome. This is why $f\big(\vartheta(x, y)\big) > 0$ for $\vartheta(x, y) > 0$ and $f\big(\vartheta(x, y)\big) < 0$ for $\vartheta(x, y) < 0$.

(b) Formulation of Regret Theory and the Maximization Problem

Suppose an agent has to choose between actions A_i and A_k, and that the outcomes of both actions in the jth state of the world are x_{ij} and x_{kj} respectively. We may define the expected modified utility of action A_i, evaluated with respect to action A_k, by

$$\mathbb{E}\left(\sum_{j=1}^{n} \psi(x_{ij}, x_{kj})\, p_j\right) = \mathbb{E}\left(\sum_{j=1}^{n} \Big(U(x_{ij}) + f\big(U(x_{ij}) - U(x_{kj})\big)\Big) p_j\right)$$

and the expected modified utility of action A_k, evaluated with respect to action A_i, by

$$\mathbb{E}\left(\sum_{j=1}^{n} \psi(x_{kj}, x_{ij})\, p_j\right) = \mathbb{E}\left(\sum_{j=1}^{n} \Big(U(x_{kj}) + f\big(U(x_{kj}) - U(x_{ij})\big)\Big) p_j\right)$$

An agent will weakly prefer A_i to $A_k \Leftrightarrow$

$$\mathbb{E}\left(\sum_{j=1}^{n} \psi(x_{ij}, x_{kj})\, p_j\right) \geq \mathbb{E}\left(\sum_{j=1}^{n} \psi(x_{kj}, x_{ij})\, p_j\right)$$

$$\Leftrightarrow \mathbb{E}\left(\sum_{j=1}^{n}\left(U(x_{ij})+f\left(U(x_{ij})-U(x_{kj})\right)\right)p_j\right) \geq \mathbb{E}\left(\sum_{j=1}^{n}\left(U(x_{kj})+f\left(U(x_{kj})-U(x_{ij})\right)\right)p_j\right)$$

$$\Leftrightarrow \mathbb{E}\left(\sum_{j=1}^{n}\left(U(x_{ij})-U(x_{kj})+f\left(U(x_{ij})-U(x_{kj})\right)-f\left(U(x_{kj})-U(x_{ij})\right)\right)p_j\right) \geq 0$$

Let $\varphi = U(x_{ij}) - U(x_{kj})$ and define a function $\phi(\varphi) = \varphi + f(\varphi) - f(-\varphi)\ \forall\ \varphi$, then $A_i \succcurlyeq A_k \Leftrightarrow$

$$\mathbb{E}\left(\sum_{j=1}^{n}\phi\left(U(x_{ij})-U(x_{kj})\right)p_j\right) \geq 0$$

Proposition 2

The function $\phi(.)$ satisfies the following properties

$\phi = 0$

$(\varphi) = -\phi(-\varphi)$, i.e.

$\phi(\varphi)$ is increasing

(φ) is convex for $\varphi \geq 0$

Proof

$\phi(0) = 0$:

Indeed, $\phi(0) = f(0) - f(-0) = 0 \Rightarrow \phi(0) = 0$

$\phi(\varphi) = -\phi(-\varphi)$:

$$\phi(-\varphi) = \phi\left(U(x_{kj})-U(x_{ij})\right) = U(x_{kj})-U(x_{ij})+f\left(U(x_{kj})-U(x_{ij})\right)-f\left(U(x_{ij})-U(x_{kj})\right)$$

$$\phi(-\varphi) = -\left(U(x_{ij})-U(x_{kj})+f\left(U(x_{ij})-U(x_{kj})\right)-f\left(U(x_{kj})-U(x_{ij})\right)\right) = -\phi(\varphi)$$

***** (φ) is increasing:

If $\varphi > 0$, then $-\varphi < 0$, $f(\varphi) > 0$ and $f(-\varphi) < 0$ as f is an increasing function. Consequently, $-f(-\varphi) > 0$ and $f(\varphi) - f(-\varphi) > 0$. This implies $\varphi + f(\varphi) - f(-\varphi) > 0$ and so $\phi(\varphi) > 0$.

If $\varphi < 0$, then $-\varphi > 0$, $f(\varphi) < 0$ and $f(-\varphi) > 0$ by assumption on f. Consequently, $-f(-\varphi) < 0$ and $f(\varphi) - f(-\varphi) < 0$. This implies that $\varphi + f(\varphi) - f(-\varphi) < 0$ and so $\phi(\varphi) < 0$.

We have $\varphi > 0 \Rightarrow \phi(\varphi) > 0$ and $\varphi < 0 \Rightarrow \phi(\varphi) < 0$. Hence $\phi(\varphi)$ must be increasing.

***** $\phi(\varphi)$ is convex for $\varphi \geq 0$:

We show that $\phi''(\varphi) \geq 0$ for $\varphi \geq 0$. Indeed, if $\varphi \geq 0$, then $\varphi \geq -\varphi$, which implies that $f(\varphi) \geq f(-\varphi)$ as f is increasing. Twice differentiating both sides of the inequality gives $f''(\varphi) \geq f''(-\varphi)$. Twice differentiating $\phi(\varphi)$ gives $\phi''(\varphi) = f''(\varphi) - f''(-\varphi)$, which therefore follows that $\phi''(\varphi) \geq 0$ since $f''(\varphi) \geq f''(-\varphi)$. Hence $\phi(\varphi)$ is convex for $\varphi \geq 0$.

To know the value of $\phi(\varphi)\ \forall\ \varphi$, it is sufficient to know the value of $\phi(\varphi) \forall\ \varphi \geq 0$. The merit of regret theory is that it is consistent with the violations of expected utility theory. It allows preferences to be intransitive, thereby capturing properties such as preference reversals. It addresses the phenomena of common ratio effect, common consequence effect (Allias paradox), reflection effect, mixed risk attitudes and two-stage gambles isolation effect. As the aim of this paper is not to discuss these implications of regret theory, we refer interested readers to the original paper on regret theory by Loomes and Sugden (1982). The paper elegantly expounds on some key implications of the theory.

Appendix 4. The Maximization Problem Setup under Regret Theory

As we have motivated in the previous appendix, regret theory rests on two fundamental assumptions. The first is that agents experience the sensations of regret and rejoicing, and the second is that agents try to anticipate and take account of these ex-post sensations when making ex-ante decisions under uncertainty. The modified utility function is therefore defined over the ex-post (final) outcomes of choices and rational investors would make choices ex-ante by maximizing the expected value of this modified utility. This allows agents to take the anticipation of regret into account in an axiomatic fashion, and is consistent with the jettisoning of both the equivalence and transitivity axioms documented in Loomes and Sugden (1982).

The modified utility function is not only defined over the outcome of the choice an agent makes, but it also includes a comparison with the outcome of another choice that could have been made in the same state of the world.

If we define the expected value of the modified utility of a rational regret averse agent faced with two choices by

$$\mathbb{E}\left(\psi(x,y)\right) = \mathbb{E}\left(U(x) + f\left(U(x) - U(y)\right)\right)$$

then the agent will seek to make a choice ex-ante that will give a final outcome x which will maximize his expected modified utility, i.e.

$$\max_x \mathbb{E}\left(\psi(x,y)\right) = \nabla$$

with $\nabla = \max_x \mathbb{E}\left(U(x) + f\left(U(x) - U(y)\right)\right)$.

It is important to note that if agents experience no regret or rejoicing at all, or if the function f is linear, then the above formulation collapses to the conventional expected utility paradigm.

Appendix 5. The Solnik Model of Portfolio Optimization within Regret Theory

A somewhat related problem to the one in this paper is the Solnik (2008) currency hedging and portfolio optimization problem within regret theory. We expound and summarize the main results as follows, with propositions and associated proofs wherever possible. Solnik (2008) considers a case in which investors can build diversified portfolios that provide better absolute performance by taking positions in foreign assets, which are considered risky, and domestic assets, which are considered non-risky. Investors are then assumed to experience a feeling of regret if their international portfolios of assets underperform their domestic portfolios of assets and, conversely, they experience a feeling of rejoicing if their international portfolios of assets overperform their domestic portfolios of assets. The domestic portfolios of assets therefore serve as a benchmark with which to compare the international portfolios of assets.

Suppose that, of their initial wealth W_0, investors allocate W_0^d to domestic assets and W_0^f to foreign assets, so that $W_0 = W_0^d + W_0^f$ represents the total wealth invested in both assets and $\frac{W_0^d}{W_0}$ and $\frac{W_0^f}{W_0}$ represents the proportions α and $(1-\alpha)$ of the total wealth invested in the domestic and foreign assets respectively. As the domestic assets serve as a benchmark, their final value W^d is nonrandom and assumed to be known in advance. The foreign assets, on the hand, are risky and their final value W^f is not known in advance. The value depends on the overall behavior of the market. If the domestic assets are denominated in US Dollars, then the dollar value of the foreign assets equals the product of the value of the foreign assets in the foreign currency and the exchange rate of the dollar to the foreign currency. This exchange rate is not known in advance, and so it is assumed to be stochastic. Also, the returns on the foreign assets are not known in advance, and so they are assumed to be stochastic. The returns are also denominated in the domestic currency i.e. dollar value, by multiplying the returns of the foreign assets in the foreign currency by the exchange rate of the dollar to the foreign currency.

Doing all these, the final dollar value W^f of the foreign assets equals the initial value W_0^f plus the returns on the initial value invested in the foreign assets RW_0^f plus the gain/loss of the initial value due to exchange rate fluctuations/currency movements sW_0^f, i.e.

$$W^f = W_0^f + RW_0^f + sW_0^f = W_0^f(1+R+s) = (1-\alpha)W_0(1+R+s)$$

and the final dollar value W of the domestic and foreign assets equals

$$W = W^d + W^f = W^d + W_0^f(1+R+s) = W^d + (1-\alpha)W_0(1+R+s)$$

where R and s are stochastic/random/non-deterministic variables denoting the return of the foreign asset in the domestic currency and the movement of the exchange rate (i.e. changes in the dollar value of the foreign currency) respectively.

As the foreign exchange rate fluctuates sporadically, i.e. can appreciate or depreciate unexpectedly without previous warning, investors decide to sell the foreign currency forward (i.e purchase a put option on the currency) as a way to hedge a proportion h of the foreign assets against currency risk. Selling a currency forward means agreeing on the exchange rate at the time the contract is entered, but carrying out the transaction at a future time. Interest rates are assumed equal globally so that the forward exchange rate (the exchange rate for the future transaction) equals the spot exchange rate (the exchange rate at the time of the contract). The argument behind this is simple. We know that the spot S and forward F exchange rates between two currencies are related to their interest rates by

$$F = S\left(\frac{1+i_\$}{1+i_C}\right)$$

and so it follows that the spot and forward exchange rates must be equal when the interest rates are equal worldwide.

Foreign assets are considered as being homogeneous and denominated in a single foreign currency C, while domestic assets are, as earlier remarked, denominated in \$. Since domestic investors view both foreign and domestic assets in terms of the domestic currency, i.e. \$, then this is equivalent to saying that domestic (American) investors care more about an appreciation of the domestic currency \$ against all other currencies. It is therefore sane to hedge against any unexpected and unfavorable movement of the dollar relative to the foreign currency.

Suppose a proportion h of the foreign assets is hedged against currency exposure, where $h \in [0,1]$. We shall call h the hedge ratio. A hedge ratio of zero implies no hedge against currency exposure and this means acomplete exposure to currency movement, while a hedge ratio of one means full hedge against currency risk and no exposure to currency movement. The participation in currency exposure is just the proportion of the foreign asset that is not hedged, i.e. $1-h$. Since h is the proportion of the foreign assets hedged against exposure to currency fluctuations, then the value of the foreign assets not subject to any currency exposure (i.e. hedged against currency exposure) is $hW_0^f s$ and so the final value of the foreign asset would be decrease by this amount, i.e.

$W = W^d + W_0^f(1+R+s) - hW_0^f s = W^d + W_0^f\big(1+R+s(1-h)\big)$. The expression reduces to the previous case of full currency exposure when $h=0$ and, when $h=1$, it reduces to the case of no currency exposure, i.e. fully hedged.

From this, the traditional utility of final wealth, as a function of the decision variable h and the two stochastic variables R and s, can be written as $U(W) = U\big(R+s(1-h)\big)$. Solnik assumes that investors exhibit regret on the decision variable h, that is, their choice to hedge a proportion h of the value of their foreign assets against currently risk. Also, an alternative decision that the investors could make would have been to select the best hedge ratio max h, from a set of other feasible hedge ratios, to give an outcome whose utility is $U(W^{\max .}) = U\left(R + \max(s(1-h))\right)$. We have the outcome of choosing a hedge ratio h and the outcome of choosing a forgone hedge ratio max h and so the modified utility can be written as

$$\psi(W, W^{\max .}) = U(R + s(1-h)) + f\left(U(R + s(1-h)) - U\left(R + \max(s(1-h))\right)\right)$$

where, as previously explained, $U(.)$ and $f(.)$ are monotonically increasing and concave; $f(.)$ is decreasingly concave, i.e. $f'' < 0$ and $f''' > 0$, and $f(0) = 0$

Proposition 3

The modified utility function $\psi(W, W^{\max .})$ can be written as

$$\psi(W, W^{\max .}) = U(R + s(1-h)) + f_{s+}\left(U(R + s(1-h)) - U(R+s)\right) + f_{s-}\left(U(R + s(1-h)) - U(R)\right)$$

Proof

We first note that the foreign currency will either appreciate or depreciate.

If the foreign currency appreciates, so that $s \in \mathbb{R}^+$, then the best forgone decision would have been to take the longest position in the foreign currency exposure and remain unhedged, i.e. $h = 0$. So, $\max(s(1-h)) = s, \forall\, s \in \mathbb{R}^+$

Similarly, if the foreign currency depreciates, so that $s \in \mathbb{R}^-$, then the best forgone alternative would have been to take the shortest position in the foreign currency and hedge completely, i.e. $h = 1$. So, $\max(s(1-h)) = 0, \forall\, s \in \mathbb{R}^-$

Accordingly, therefore, for any $\in \mathbb{R}^- \cup \mathbb{R}^+$, we have

$$\psi(W, W^{\max .}) = U(R + s(1-h)) + f_{s+}\left(U(R + s(1-h)) - U(R+s)\right) + f_{s-}\left(U(R + s(1-h)) - U(R)\right)$$

Proposition 4

The impact of a currency movement s is such that regret aversion induces currency loss aversion, i.e. a depreciation of the foreign currency leads to reductions in financial wealth and investors are more sensitive to these reductions than to corresponding increases in financial wealth.

Proof

We take the current exchange rate movement $s = 0$ as a reference point. In order to prove that regret aversion induces currency loss aversion, we must prove that

$$\frac{\partial \psi}{\partial s}\Big|_{s \in \mathbb{R}^-} > \frac{\partial \psi}{\partial s}\Big|_{s \in \mathbb{R}^+}$$

Now, taking the reference point into account, the left and right derivatives of the modified utility function with respect to s are given by

$$\frac{\partial \psi}{\partial s}\Big|_{s \in \mathbb{R}^-} = (1-h)U'(R) + (1-h)f'(0)U'(R)$$
$$\frac{\partial \psi}{\partial s}\Big|_{s \in \mathbb{R}^+} = (1-h)U'(R) - hf'(0)U'(R)$$

So,

$$\frac{\partial \psi}{\partial s}\Big|_{s \in \mathbb{R}^-} - \frac{\partial \psi}{\partial s}\Big|_{s \in \mathbb{R}^+} = f'(0)U'(R) > 0$$

since f and U are monotonically increasing . Thus, we have

$$\frac{\partial \psi}{\partial s}\Big|_{s \in \mathbb{R}^-} > \frac{\partial \psi}{\partial s}\Big|_{s \in \mathbb{R}^+}$$

as required to prove.

We have shown that, taking the current exchange rate movement as a reference point, investors are more sensitive to reductions in financial wealth, i.e. $\frac{\partial \psi}{\partial s} > 0$ for $s \in \mathbb{R}^-$, than to corresponding increases in financial wealth, i.e. $\frac{\partial \psi}{\partial s} < 0$ for $s \in \mathbb{R}^+$. We remark that $s \in \mathbb{R}^-$ indicates a depreciation of the foreign currency, resulting in financial wealth reductions while $s \in \mathbb{R}^+$ indicates an appreciation of the foreign currency, resulting in financial wealth increases. Hence, regret aversion induces loss aversion.

Proposition 5

The modified utility function $\psi(.)$ is concave with respect to s, $\forall\, s \in \mathbb{R}$

Proof

The proof is in two phases. First, we prove $\frac{\partial^2 \psi}{\partial s^2} < 0\ \forall\, s > 0$. Second, we prove $\frac{\partial^2 \psi}{\partial s^2} < 0\ \forall\, s < 0$.

Let $\theta = 1 - h$, then

$$\psi(W, W^{\max .}) = U(R + s\theta) + f_{s+}\left(U(R + s\theta) - U(R+s)\right) + f_{s-}\left(U(R + s\theta) - U(R)\right)$$

For any positive values of s, the first and second derivatives of ψ are given by

$$\frac{\partial \psi}{\partial s} = \theta U'(R + s\theta) + \left(\theta U'(R + s\theta) - U'(R+s)\right)f'\left(U(R + s\theta) - U(R+s)\right)$$
$$\frac{\partial^2 \psi}{\partial s^2} = \theta^2 U''(R + s\theta) + \left(\theta^2 U''(R + s\theta) - U''(R+s)\right)f'\left(U(R + s\theta) - U(R+s)\right)$$
$$+ \left(\theta U'(R + s\theta) - U'(R+s)\right)^2 f''\left(U(R + s\theta) - U(R+s)\right)$$

We know that $\theta^2 U''(R + s\theta) < 0$, $f'\left(U(R + s\theta) - U(R+s)\right) > 0$, $\left(\theta U'(R + s\theta) - U'(R+s)\right)^2 > 0$, $f''\left(U(R + s\theta) - U(R+s)\right) < 0$ and $\left(\theta U'(R + s\theta) - U'(R+s)\right)^2 f''\left(U(R + s\theta) - U(R+s)\right) < 0$ Thus, to prove that $\frac{\partial^2 \psi}{\partial s^2} < 0$, we only need to show that $\left(\theta^2 U''(R + s\theta) - U''(R+s)\right)f'\left(U(R + s\theta) - U(R+s)\right) < 0$ and to do this, we need to show that $\left(\theta^2 U''(R + s\theta) - U''(R+s)\right) < 0$ since we already know that $f'\left(U(R + s\theta) - U(R+s)\right) > 0$.

Now, for any positive values of s and for $\theta = 1 - h \leq 1$, we have $s\theta < s$ and $R + s\theta \leq R + s$. Furthermore, as investors prefer more to less, we have $U(R + s\theta) \leq U(R + s)$. Twice differentiating both sides of the inequality with respect to s gives $\theta^2 U''(R+s\theta) \leq U''(R+s)$ or $(1-h)^2 U''(R+s\theta) \leq U''(R+s)$. Complete expansion gives $(1-2h+h^2)U''(R+s\theta) \leq U''(R+s)$. Rearranging terms gives $U''(R+s\theta) - U''(R+s) \leq h(2-h)U''(R+s\theta) = (1-\theta)(1+\theta)U''(R+s\theta) \leq 0 \quad \Rightarrow \quad U''(R+s\theta) - U''(R+s) \leq 0$. Consequently, for positive values of s, we have $\frac{\partial^2\psi}{\partial s^2}<0$ as required.

The second phase of the proof proceeds in a similar fashion. For any negative values of s, the first and second derivatives of ψ are given by

$$\frac{\partial\psi}{\partial s} = \theta U'(R+s\theta) + \theta U'(R+s\theta) f'\big(U(R+s\theta) - U(R)\big)$$

$$\frac{\partial^2\psi}{\partial s^2} = \theta^2 U''(R+s\theta) + \big(\theta U'(R+s\theta)\big)^2 f''\big(U(R+s\theta)-U(R)\big) + \theta^2 U''(R+s\theta) f'\big(U(R+s\theta)-U(R)\big)$$

The second derivative is trivially negative because $\theta^2 U''(R+s\theta) < 0$, $f''\big(U(R+s\theta)-U(R)\big) < 0$ and $\big(U(R+s\theta)-U(R)\big) > 0$. Consequently, for negative values of s, we have $\frac{\partial^2\psi}{\partial s^2}<0$ as required. Thus, for all values of h and R, $\frac{\partial^2\psi}{\partial s^2} < 0 \ \forall\ s > 0$ and $\frac{\partial^2\psi}{\partial s^2}<0 \ \forall\ s < 0$ and so ψ is continuous and concave with respect to s. Still on Solnik model/problem, we shall now discuss how the optimization problem is set up in a regret theoretic framework and how optimal investment rules are derived.

Solnik Model: Problem Setup and Results of Portfolio Optimization

***** *Problem Setup*

The problem is structured in such a way as to obtain the optimal hedge ratio h^* by maximizing the expected modified utility with respect to the decision variable h. The expected modified utility is given by

$$\mathbb{E}[\psi(W, W^{\max\cdot})] = \mathbb{E}[U(R+s\theta)] + \mathbb{E}_{s+}\big[f\big(U(R+s\theta) - U(R+s)\big)\big] + \mathbb{E}_{s-}\big[f\big(U(R+s\theta)-U(R)\big)\big]$$

and investors obtain the optimal hedge ratio by solving

$$\max_{0\leq\theta=1-h\leq 1} \mathbb{E}[U(R+s\theta)] + \mathbb{E}_{s+}\big[f\big(U(R+s\theta) - U(R+s)\big)\big] + \mathbb{E}_{s-}\big[f\big(U(R+s\theta)-U(R)\big)\big]$$

where $\mathbb{E}_{s+}$ and $\mathbb{E}_{s-}$ denote the expectation taken over positive and negative values of s respectively.

Proposition 6a

If $f(.)$ is nonlinearly concave, then the expected modified utility $\mathbb{E}[\psi(W, W^{\max\cdot})]$ is concave in h and so the first order condition $\frac{\partial\mathbb{E}[\psi(W,W^{\max\cdot})]}{\partial h}=0$ is necessary and sufficient for optimality.

Remark

Except for very particular and simplistic functions $U(.)$ and $f(.)$, the first order condition is not explicitly solvable in h and applying numerical solutions results in little generalization of the hedging rules. To unravel this challenge, Solnik make a Taylor expansion of $\mathbb{E}[\psi(W, W^{\max\cdot})]$ and apply the two-moment approximation of Arrow Pratt. The resulting expression is then maximized with respect to h , giving explicit hedging rules. The two-moment approximation of Arrow Pratt is very crucial because it helps to explicitly retain the parameters of the modified utility in the optimal hedging rules derived from the maximization of the expected modified utility, something that cannot be achieved/done by simply assuming normality of returns, R and s.

***** *Key results of Solnik Model*

The results of the optimization problem for different assumptions on the exchange rate movement are summarized in the propositions below

Proposition 6b (Pure Currency Risk Minimization)

Suppose that investors have no prior information on expected currency returns or correlation between asset and currency returns (This is called pure risk, which implies no currency risk premium). If we assume that the distribution of currency returns is symmetric and the return on foreign assets is non-stochastic, then the optimal hedge ratio for a regret averse investor is given by

$$h^* = 1 - \frac{1}{2}\frac{\rho}{\lambda+\rho}$$

where $\lambda = -\frac{U''}{U'} > 0$ is traditional risk aversion and $\rho = -\frac{U'f''}{1+f'} > 0$ is regret aversion

Remark

In the absence of regret, the function f is either linear, zero or constant, and so $f'' = 0$ which then implies that $\rho = 0$ and so the optimization problem reduces to the expected utility case. Consequently, the optimal hedge ratio $h^* \ \forall\ \lambda \in \mathbb{R}$ is given by $h^* = 1$, i.e. full currency hedging and no participation in currency exposure. This is the optimal hedge ratio for a traditional expected utility investor who considers only the outcome of his choice and thus feels no regret. This is a typical complete-hedging risk minimization result.

Since regret aversion ρ and risk aversion λ are both positive, then $0 < \frac{\rho}{\lambda+\rho} <1$ and so $\frac{1}{2} < h^* <1$. This shows that the optimal hedge ratio is always between 50% and 100%.

Because $\frac{\partial h^*}{\partial\rho} = -\frac{1}{2}\frac{\lambda}{(\lambda+\rho)^2} <0$, the lower the regret aversion ρ the higher the optimal hedge ratio and the lower the participation in currency exposure.

When regret aversion ρ is sufficiently larger than risk aversion λ, we have $\lambda + \rho \approx \rho$ and $h^* \approx \frac{1}{2}$, i.e. the optimal hedge ratio gets close to 50% when investors are infinitely regret averse. When it is smaller, the optimal hedge ratio tends to 100% as ρ nears zero.

For sufficiently large regret aversion, investors care exclusively about the level of regret associated with any hedging decisions. For instance, when investors try to hedge fully, they anticipate the maximum regret associated with a strong appreciation of the foreign currency. This anticipation is so high that it forces them to reject such a hedging decision. Conversely, when investors try not to hedge at all but instead participate fully in currency exposure, they anticipate the high regret associated with a depreciation of the foreign currency and this coerces them to avoid taking such a hedging decision.

Proposition 7 (General case)

If the returns R and s on foreign assets and currency are both stochastic, and if the expected currency return $\mathbb{E}(s)$ is nonzero (existence of currency risk premium and no pure risk) so that there exists a correlation between asset and currency returns, then the optimal hedge ratio for a regret averse investor, in the case where the distribution of currency returns is not necessarily symmetric, is given by

$$h^* = 1 - \frac{\sum_{s+} \cdot}{\sum_{s} \cdot}\frac{\rho}{\rho+\lambda} - \frac{\mathbb{E}(s)}{(\rho+\lambda)\sum_{s}\cdot} + \frac{\mathbb{cov}(R,s)}{\sum_{s}\cdot}\frac{\lambda}{\lambda+\rho}$$

where $\sum_{s}\cdot = \mathbb{E}(s^2)\ \forall\, s \in \mathbb{R}$ and $\sum_{s+} = \mathbb{E}(s^2)\ \forall\, s \in \mathbb{R}^+$. The regret, speculative and covariance terms are $-\frac{\sum_{s+}\cdot}{\sum_{s}\cdot}\frac{\rho}{\rho+\lambda}$, $-\frac{\mathbb{E}(s)}{(\rho+\lambda)\sum_{s}\cdot}$ and $\frac{\mathbb{cov}(R,s)}{\sum_{s}\cdot}\frac{\lambda}{\lambda+\rho}$ respectively.

Remark

In the absence of regret aversion, i.e. $\rho = 0$ and the optimal hedge ratio for an expected utility investor under the above assumption is given by $h^* = 1 - \frac{\mathbb{E}(s)}{\lambda\sum_{s}\cdot} + \frac{\mathbb{cov}(R,s)}{\sum_{s}\cdot}$. From this, we see that a positive expectation on the foreign currency movement reduces the optimal hedge ratio and increases participation in currency exposure. The lower the risk aversion, the more investors speculate and therefore the lower the hedge ratio. In the same vein, a negative correlation between foreign currency movement and asset returns reduces the optimal hedge ratio.

In this case where the return distribution is not necessarily symmetric and where the expected currency return differs from/ is not zero, the term $\sum_{s+}$ will generally be different from the term $\frac{1}{2}\sum_{s}\cdot$ To see this, we need to first recognize that $\sum_{s} = \sum_{s+} + \sum_{s-}$. Now, if the expected currency return is more than zero (investors anticipate an appreciation of the foreign currency), then $\sum_{s+} > \sum_{s-}$ and $\sum_{s+} > \sum_{s} - \sum_{s+} \Rightarrow \sum_{s+} > \frac{1}{2}\sum_{s}\cdot$ Plugging this into the optimal hedge ratio h^* shows that regret averse investors will hedge less and participate more in foreign currency exposure because they anticipate to experience less regret. On the other hand, if the expected currency return is less than zero (investors anticipate a depreciation of the foreign currency), then $\sum_{s+} < \frac{1}{2}\sum_{s}\cdot$ Plugging this into the optimal hedge ratio h^* shows that regret averse investors will hedge more and participate less in foreign currency exposure because they anticipate to experience more regret if the foreign currency depreciates. Conjecturally, a similar result holds true for a skewed distribution. A distribution that is positively skewed implies that there are significantly large currency returns with low probability of occurring. Regret averse investors will therefore hedge less and participate more in currency exposure so as to take advantage of these large returns even if they know that the returns have a small chance of occurring/ a rare event. Therefore, positive skewness leads to taking a huge position in currency exposure (hedging less) while negative skewness leads to a lower participation in currency exposure (hedging more). This helps to reduce the potential for regret.

Looking at the optimal hedge ratio from the point of view of the speculative term, we see that a positive expectation on the foreign currency movement increases the participation in foreign currency exposure and reduces the optimal hedge ratio. Regret averse investors tend to speculate less on their anticipations of currency movements.

Now that we have successfully discussed the portfolio optimization and hedging model of Solnik, we next proceed to the main objective of this chapter which is to perform the analysis of a pension fund in a regret theoretic framework.

Appendix 6. Proof of Concavity and Monotonicity of the Value Function

Under all the suitably stated conditions that must hold for the optimization problem of the pension fund, the **modified utility function, and hence the value function, is increasing and concave in W.**

Proof

We have

$$\frac{\partial\psi}{\partial W} = \big(W(t) - M(t)\big)^{-\vartheta}\left(1 + \rho\left(\frac{1}{1-\vartheta}\Big(\big(W(t)-M(t)\big)^{1-\vartheta} - \big(W^o(t) - M(t)\big)^{1-\vartheta}\Big) + a\right)^{\rho-1}\right) > 0$$

and

$$\frac{\partial^2\psi}{\partial W^2} = -Q < 0$$

where

$$Q = \left[\begin{Bmatrix} \vartheta\big(W(t)-M(t)\big)^{-\vartheta-1}\left(1 + \rho\left(\frac{1}{1-\vartheta}\Big(\big(W(t)-M(t)\big)^{1-\vartheta} - \big(W^o(t)-M(t)\big)^{1-\vartheta}\Big) + a\right)^{\rho-1}\right) \\ + \\ \rho(1-\rho)\left(\frac{1}{1-\vartheta}\Big(\big(W(t)-M(t)\big)^{1-\vartheta} - \big(W^o(t)-M(t)\big)^{1-\vartheta}\Big) + a\right)^{\rho-2}\big(W(t)-M(t)\big)^{-2\vartheta} \end{Bmatrix}\right] > 0$$

Hence the modified utility is concave in W, implying that the first order condition is necessary for optimality. This is a standard result in optimization theory. It guarantees that, under suitable conditions for the optimization problem, the value function is increasing and concave in W. Here the value function is increasing and concave in W because it is a function of the modified utility, which is itself increasing and concave in W.

Appendix 7. Explanation of Important Concepts

Here we present the notion of financial markets in relation to the pension fund industry and describe some relevant concepts that will make clear the arguments presented in the paper.

Financial Markets: Financial markets serve as a transaction point where investors trade securities, commodities and other transposable financial instruments such as currencies and derivatives at prices that reflect demand and supply. Investors can be private and institutional investors while financial markets can be capital, money, derivatives, commodities, foreign exchange and insurance markets.

Private and Institutional Investors: Private investors are individuals who participate in transaction activities in the financial markets with limited initial capital while institutional investors comprise large corporations that engage in buy-side deals with large initial capital and restricted protective regulations. Institutional investors account for a majority of overall volume of trades in the financial markets and are known to include pension funds, hedge funds, insurance companies, brokerages, mutual funds, investment banks and asset management firms etc. We dwell mainly on the investment strategy of pension funds institutional investors because that is the basis of this research.

Pension Funds: Pension funds are important institutional investors that provide retirement income and benefits to their clients/subscribers. Their presence is especially felt in the money and capital markets where sell-side investors such as private and public enterprises as well as governments come to raise short- and long-term funds to finance business operations and capital expenditures. Pension funds worldwide hold over $20 trillion in assets and therefore dominate other institutional investors in terms of investments in assets [10]. This suggests the importance of pension funds and hence the need for them to be studied.

Researchers have studied two different types of pension plans, which are:

* Defined-benefit plans
* Defined-contribution plans
* Defined-Benefit and Defined-Contribution Plans

While defined-benefit pension plans are employer-sponsored plans in which a retired employee receives specific retirement benefits at retirement based on years of service and salary history, defined- contribution pension plans, on the other hand, allow the employee to make seasonal contributions to the pension fund before retirement, but there is no way the employee can know the specific retirement benefits he will enjoy at retirement because the fund is invested and everything depends on the rate of return of the invested fund. Another important distinction between both pension plans is that, in the case of defined-contribution pension plans, the employee sets up an investment account and almost solely makes the entire investment decisions while, in the case of defined-benefit pension plans, the employer makes all investment decisions and has the final say over the invested fund. Thus, the employee bears the investment risk in defined-contribution pension plans while the employer manages the investment portfolio and bears the investment risk in the case of defined-benefit plans. Examples of these plans are annuities and 401k plans. Annuities are defined-benefit plans that provide fixed monthly payments to each employee at retirement, while 401k plans are defined-contribution plans that allow tax-deferred income to finance retirement benefits.

Unlike other institutional investors, the analysis of pension funds requires the introduction of three unique characteristics:

* The behavior of the fund wealth in the accumulation phase (Ac)
* The variation of the fund wealth in the decumulation phase (Dc)
* The mortality risk of the subscriber

Accumulation and Decumulation Phases: The representative subscriber makes contributions to the pension fund in the accumulation phase and so the fund wealth swells. In the decumulation phase, the pension fund makes mandatory payments (pensions) to the subscriber and so the fund wealth shrinks.

Researchers and practitioners have established the existence of a link between contributions in the accumulation phase and pensions in the decumulation phase for both defined-benefit and defined-contribution plans. In defined-benefit plans, the employer fixes benefits in advance and contributions are designed to maintain the fund in balance. In defined-contribution plans, contributions are fixed but benefits depend on the returns of the invested funds. The model presented here concerns the case of a pension fund that offers its subscribers a deterministic pension plan. The deterministic pension plan is such that the subscriber makes a constant contribution to the fund while the pension fund pays a constant pension to the subscriber at retirement. This is the so called ‘Cash Balance Plan’, which is especially prominent in the US.

Cash balance plans are fundamentally defined-benefit pension plans that operate as defined-contribution pension plans in that they require an employee to set up and maintain an investment account, while the investment account earns a fixed rate of return that may change over time. Thus, cash balance plans share the characteristics of both defined-benefit and defined-contribution pension plans because they require an employee to have an investment account, which makes them defined-contribution, and the investment account earns a fixed rate of return, which makes them defined-benefit.

Mortality Risk: In any pension plan, mortality risk is the risk that an active subscriber who is accumulating his pension benefits will die earlier than expected. This contrasts longevity risk, which is the risk that an inactive member with pensions in payment will live longer than expected. Essentially, mortality risk is restricted to accumulation phase while longevity risk is restricted to decumulation phase. As we can quickly infer therefore, longevity risk is worse than mortality risk because it ultimately leads to the depletion of the wealth of the pension fund if it carries on for a very long time.

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