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An expanded multiplier-accelerator model

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Abstract. This paper revisits the standard multiplier-accelerator model, as advanced by Samuelson. While borrowing on the main assumptions of the multiplier-accelerator, we check the validity of Keynesian theory. Using higher-order difference equations and advanced-level mathematical techniques we solve the tax-augmented multiplier-accelerator model, as well as the open economy one. We find that the values of equilibrium national income are identical to the simple national-income model in the absence of the accelerator. We solve the simple multiplier-accelerator model both in present terms and withprolonged consumption. We solve for equilibrium consumption, tax, and imports which are unaffected by the accelerator. All results conform to Keynesian theory where investment, government spending and exports have a favorable multiplying effect on national income through their respective multipliers. The accelerator coefficient affects neither those multipliers, nor the income and the non-income tax multipliers. Expanding the multiplier-accelerator by the volume of foreign trade, taxation or both does not change the values of Keynesian variables. Adding an accelerator leaves optimal values unaffected but, more importantly, reinforces Keynesian theory.
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1. Introduction

Harrod (1936, 1939) was the first British economist to analyze the interplay between the multiplier and the accelerator as part of business cycle theory. Harrod intensively exchanged ideas with Keynes before the publication of the General Theory and his own Trade Cycles. Samuelson (1939) advanced a rigorous model of the interaction between the multiplier and the accelerator setting the foundation of macrodynamics. Samuelson aimed at integrating the principle of the multiplier, a newly proposed concept in the Keynesian theory of national income determination at the time, with the older concept of the accelerator in a single theory. The monetarist school criticizes the concept of the investment multiplier in that investment is simply a more variable component than consumption and there is no empirical evidence that they

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are affected by common shocks or that investment and consumption affect each other at all (Friedman & Schwartz 1963). Some other criticisms involve the endogeneity principle which guides the expectations within the economy but is unable to produce long lasting fluctuations and thus cannot be relied on in constructing long-term predictions. Empirical observations for the parameters of national income have shown that its trajectory is unstable. There are numerous macrodynamic models in the literature and many of them focus on general-equilibrium conditions for an open economy (Harris, 1984; Bruno & Sachs, 1985 and Devarajan & Go, 1998). None of these models is based entirely on the multiplier-accelerator framework and its direct applicability. Some more recent studies of the multiplier-accelerator process involve Puu, Gardini & Sushko (2005), Westerhoff (2006), Matsumoto & Szidarovszky (2015).

In combining the multiplier and the accelerator in the national economy many believe that the two complement each other and should be studied simultaneously. A natural question to ask, though, is which effect dominates in the context of Samuelson's simple multiplier-accelerator setting. Does investment drive national income, as consistent with Keynesian beliefs, or on the contrary, investment results from firms' expectations about the growth of domestic output and generates further output. While the two processes are endogenously related and nurture each other, it is legitimate to ask which is the leading one in the economy. The purpose of this paper is to check the validity of Keynesian theory related to the investment and government spending multiplier, as formulated by Keynes, and to evaluate the role of the accelerator coefficient in the standard combined model, as advanced by Samuelson. We do not essentially introduce a new theory, a general equilibrium representation or a business cycle interpretation. Our goal is to check standard theory by use of advanced discrete time, dynamic analysis and higher-order difference equations. Solving the model in various contexts and using advanced-level mathematical techniques, we do not find strict confirmation that the accelerator effect prevails over the multiplier effect. We find that at the steady state, all equilibrium results of the Keynesian national-income model hold. The values of equilibrium national income are identical to those under Keynesian assumptions in the case of the simple national-income model and multiplier in the absence of the accelerator, the tax-augmented national-income model as well as the national-income model in the conditions of an open economy, that is, with foreign trade added. We solve the simple multiplier-accelerator model both in present terms and by movingconsumption one period backward. We solve for equilibrium consumption, tax, and imports. All results conform to Keynesian theory where investment, government spending and exports have a favorable multiplying effect on national income through the investment, government spending and exports multipliers, respectively. We obtain two more multipliers in the multiplier-accelerator setting, namely, the income and the non-income tax multipliers. Their values are identical to those of Keynesian economics. The analysis leads us to believe that adding an accelerator coefficient reaffirms Keynesian findings, reinforcing thus the validity of the theory.

2. The standard multiplier-accelerator relationship

The traditional multiplier-accelerator model shows a dual-causality relationship between aggregate investment and national income. In the presence of positive exogenous shocks increased investment has a multiplying effect on national income by the amount of the investment multiplier but the increase in income makes firms believe that demand for their goods has increased. This stimulates firms to invest more in capital stock, which is known as the accelerator principle (Samuelson, 1939). Thus, investment stimulates national income through the multiplier process while national incomeincreases investment through the accelerator process, and they affect each other in an interactive way. The effect of a downturn in the economy would be adverse on both national income and investment which makes the multiplier-accelerator process relevant to the business cycle. In a recession the multiplier-accelerator process would force the economy to contract. The standard model assumes the following three equations

$$Y_t = C_t + I_t + G_o \tag{1}$$
$$C_t = \beta Y_{t-1} \qquad 0 < \beta < 1 \tag{2}$$
$$I_t = \alpha(C_t - C_{t-1}) \qquad \alpha > 0 \tag{3}$$

where national income depends on current consumption, investment and government spending. Government spending G_o is presumed to be exogenous. In this simple model people spend based on income earned in the previous period where β shows the share of income that is consumed, that is, the marginal propensity to consume. Investment is positively related to the increase in aggregate consumption $\Delta C_{t-1} = C_t - C_{t-1}$, as shown by the accelerator coefficient α. The investment equation implies that the increased consumption makes firms optimistic, seeing demand for their product rise and, thereof, nurtures them to increase investment. We substitute the respective terms for C_t in equation (3).

$$I_t = \alpha\beta(Y_{t-1} - Y_{t-2}) \tag{4}$$

Substitutingfor consumption and investment into (1)

and moving forward by two periods gives a second-order difference equation for national income

$$Y_{t+2} - \beta(1+\alpha)Y_{t+1} + \alpha\beta Y_t = G_o \tag{5}$$

The parameters here are $a_1 = -\beta(1+\alpha)$, $a_2 = \alpha\beta$ and $c = G_o$, whereas the particular integral is

$$\bar{Y} = \frac{c}{1+a_1+a_2} = \frac{G_o}{1-\beta(1+\alpha)+\alpha\beta} = \frac{G_o}{1-\beta} \tag{6}$$

Given that the marginal propensity to consume β is less than 1, we obtain a meaningful intertemporal equilibrium for national income which is positively related to exogenous government spending. Furthermore, $\frac{1}{1-\beta}$ is the value of the multiplier. The characteristic equation of the model is

$$b^2 - \beta(1+\alpha)b + \alpha\beta = 0, \tag{7}$$

giving the roots

$$b_{1,2} = \frac{\beta(1+\alpha) \pm \sqrt{\beta^2(1+\alpha)^2 - 4\alpha\beta}}{2} \tag{8}$$

By Viete's formula it follows that the two roots satisfy the conditions

$$b_1 + b_2 = \beta(1+\alpha) \tag{9}$$
$$b_1 b_2 = \alpha\beta \tag{10}$$

Based on these results, we conclude that

$(1-b_1)(1-b_2) = 1-(b_1+b_2)+b_1b_2 = 1-\beta(1+\alpha)+\alpha\beta = 1-\beta$, and, hence, (11)

$$0 < (1-b_1)(1-b_2) < 1 \tag{12}$$

For the complementary function in the complete solution we have three possible cases depending on whether $a_1^2 > 4a_2$ from the characteristic equation. This first case is equivalent to

$$\beta^2(1+\alpha)^2 > 4\alpha\beta \tag{13}$$
$$\beta > \frac{4\alpha}{(1+\alpha)^2} \tag{14}$$

Both distinct real roots are positive since $b_1b_2 > 0$ and $b_1 + b_2 > 0$. This precludes oscillation according to the theory of second-order difference equations and convergence to the intertemporal equilibrium would depend on whether b_1 and b_2 are fractions. Several cases might be considered but the legitimate ones are presented in Table 1. Similar is the case of a repeated root where $b = \frac{\beta(1+\alpha)}{2}$, again with a positive sign since both α and β are positive parameters. There is no oscillation again and the dynamic stability of national income depends on whether the repeated root is a fraction. In the final case of conjugate complex roots the presence of $R = \sqrt{a_2} = \sqrt{\alpha\beta}$ determines stepped fluctuation. If $R < 1$, the fluctuation would be narroweddown, while for $R \geq 1$ we would have explosive growth.

Table 1. *Possible values for the roots of national income*

Case	Subcase	Value of $\alpha\beta$	Time path of Y_t	Dynamic stability
1. Distinct real roots				
$\beta > \frac{4\alpha}{(1+\alpha)^2}$	$0 < b_1 < b_2 < 1$	$\alpha\beta < 1$	Nonoscillatory	Convergence
$\beta > \frac{4\alpha}{(1+\alpha)^2}$	$1 < b_1 < b_2$	$\alpha\beta > 1$	Nonoscillatory	Divergence
2. Repeated real roots				
$\beta = \frac{4\alpha}{(1+\alpha)^2}$	$0 < b < 1$	$\alpha\beta < 1$	Nonoscillatory	Convergence
$\beta = \frac{4\alpha}{(1+\alpha)^2}$	$b > 1$	$\alpha\beta > 1$	Nonoscillatory	Divergence
3. Complex roots				
$\beta < \frac{4\alpha}{(1+\alpha)^2}$	$R < 1$	$\alpha\beta < 1$	Stepped fluctuation	Convergence
$\beta < \frac{4\alpha}{(1+\alpha)^2}$	$R \geq 1$	$\alpha\beta \geq 1$	Stepped fluctuation	Divergence

We can summarize that the time path of national income is convergent only if $\alpha\beta < 1$. National income can have cyclical fluctuations endogenously without any external shocks and merely due to the interactive play of the multiplier and the accelerator process. We further extend the period of investment. Not only would present consumption depend on national income in the previous period but current investment depends on the change in consumption in the previous period $\Delta C = C_{t-1} - C_{t-2}$, rather than $\Delta C = C_t - C_{t-1}$.

$$Y_t = C_t + I_t + G_o \tag{15}$$
$$C_t = \beta Y_{t-1} \qquad 0 < \beta < 1 \tag{16}$$
$$I_t = \alpha(C_{t-1} - C_{t-2}) \qquad \alpha > 0, \tag{17}$$

where upon substitution for consumption in (17) and consequently in (15), we obtain a third-order difference equation in national income alone.

$$Y_{t+3} - \beta Y_{t+2} - \alpha\beta Y_{t+1} + \alpha\beta Y_t = G_o \tag{18}$$

This again generates the standard multiplier, since for the particular integral we have the same result as in (6)

$$\bar{Y} = \frac{c}{1+a_1+a_2} = \frac{G_o}{1-\beta-\alpha\beta+\alpha\beta} = \frac{G_o}{1-\beta} \tag{19}$$

The choice of period for the increase in consumption that stimulates investment, i.e. the period of the acceleration process, does not affect the result and the standard multiplier obtains. The intertemporal equilibrium of aggregate consumption remains unchanged, too. The standard Samuelson model produces a second-order difference equation for aggregate consumption.

$$C_{t+2} - \beta(1+\alpha)C_{t+1} + \alpha\beta C_t = \beta G_o \tag{20}$$

Hence, for equilibrium aggregate consumption,

$$\bar{C} = \frac{\beta G_o}{1-\beta} = \beta\bar{Y} \tag{21}$$

Moving the change in consumption a period backward such that the new model is

$$Y_t = C_t + I_t + G_o \tag{22}$$

$$C_t = \beta Y_{t-1} \qquad 0<\beta<1 \tag{23}$$

$$I_t = \alpha(C_{t-1} - C_{t-2}) \qquad \alpha>0 \tag{24}$$

produces a third-order difference equation solely in consumption with the same equilibrium value.

$$C_{t+3} - \beta C_{t+2} - \alpha\beta C_{t+1} + \alpha\beta C_t = \beta G_o \qquad \bar{C} = \frac{\beta G_o}{1-\beta} = \beta\bar{Y} \tag{25}$$

Comparing the discrete and the continuous time outcome by transforming the standard multiplier-accelerator model from a discrete into a continuous time form,

$$Y(t) = C(t) + I(t) + G_o \tag{26}$$

$$C(t) = \beta Y(t) \qquad 0<\beta<1 \tag{27}$$

$$I(t) = \alpha\frac{dC}{dt} \qquad \alpha>0 \tag{28}$$

where $Y(t), C(t)$ and $I(t)$ are continuous and differentiable functions of time. Hence, the normalized solution for national income is

$$Y'(t)-\frac{(1-\beta)}{\alpha\beta}Y(t)=-\frac{G_o}{\alpha\beta} \tag{29}$$

giving the time path of equilibrium national income as

$$Y(t)=\left[Y(0)-\frac{G_o}{1-\beta}\right]e^{\frac{1-\beta}{\alpha\beta}}+\frac{G_o}{1-\beta} \tag{30}$$

This gives the standard multiplier again where $\bar{Y}=\frac{G_o}{1-\beta}$. The time path of national income is divergent from this intertemporal equilibrium level since the values of the parameters determine an explosive growth and a positive exponent $\frac{(1-\beta)}{\alpha\beta}$ where national income starts from an initial level $Y(0)$. We see that the equilibrium value of national income is identical in the continuous and the discrete case. The value of the accelerator does not affect equilibrium national income, nor the value of the multiplier, however, the accelerator does play a role in the explosiveness of the time path, that is, how quickly national income diverges from the initial equilibrium level. Since the accelerator is in the denominator of the exponential term in the complementary function, a higher value of the accelerator α (greater scaling up effect of national income on aggregate investment) slows down the deviation of national income from its equilibrium level.

3. An expanded multiplier-accelerator model with taxation added

In his original theory Keynes included two types of tax, income and non-income, and investigated their effect on national income through the two multipliers, an income tax and a non-income tax multiplier, respectively. Keynes found the effect of any kind of tax, direct or indirect, to be negative on national income, although in the presence of a government sector and government expenditure, the role of tax collection would be unavoidable. It, therefore, seems reasonable to verify the effect of those two types of taxes in the combined multiplier-accelerator model. We follow the simple setting of the national-income model where non-income tax is independent of national income while income tax is a share of it. The model with taxation has four main assumptions, as presented below,

$$Y_t=C_t+I_t+G_o \tag{31}$$

$$C_t=\beta(Y_{t-1}-T_{t-1}) \qquad 0<\beta<1 \tag{32}$$

$$I_t=\alpha(C_t-C_{t-1}) \qquad \alpha>0 \tag{33}$$

$$T_t=\gamma+\delta Y_t \qquad \gamma>0 \quad 0<\delta<1 \tag{34}$$

whereγ is non-income tax and δ is the income-tax rate. Substituting (32) and consequently (34) into (33),

$$I_t = \alpha\beta(Y_{t-1} - \gamma - \delta Y_{t-1} - Y_{t-2} + \gamma + \delta Y_{t-2}) \tag{35}$$

$$I_t = \alpha\beta(1-\delta)(Y_{t-1} - Y_{t-2}) \tag{36}$$

Substituting this result in (31) and moving the equation two periods forward,

$$Y_{t+2} - \beta(1-\delta)(1+\alpha)Y_{t+1} + \alpha\beta(1-\delta)Y_t = G_o - \beta\gamma \tag{37}$$

At the steady state we have $\bar{Y} = Y_t = Y_{t+1} = Y_{t+2}$ so for equilibrium national income $\bar{Y}$ we have

$$\bar{Y} - \beta(1-\delta)(1+\alpha)\bar{Y} + \alpha\beta(1-\delta)\bar{Y} = G_o - \beta\gamma \tag{38}$$

$$\bar{Y} = \frac{G_o - \beta\gamma}{1-\beta(1-\delta)(1+\alpha)+\alpha\beta(1-\delta)} = \frac{G_o - \beta\gamma}{1-\beta(1-\delta)} \tag{39}$$

which is the value of equilibrium national income. It is expected that government spending G_o exceeds a minimum threshold level of $\beta\gamma$ for a positive national income to obtain. In the absence of taxation, that is, when income and non-income tax areassumed to be zero, the value of national income is exactly equal to that under the simple national income, or $\bar{Y} = \frac{G_o}{1-\beta}$. The accelerator α plays no role in the multiplier again, as in the case of Samuelson's ordinary multiplier-accelerator model. The denominator being a fraction gives rise to a ratio bigger than 1.

$$\frac{\Delta\bar{Y}}{\Delta G_o} = \frac{1}{1-\beta(1-\delta)} > 1 \tag{40}$$

We get two more standard multipliers, the non-income tax multiplier and the income tax multiplier

$$\frac{\Delta\bar{Y}}{\Delta\gamma} = -\frac{\beta}{1-\beta(1-\delta)} < 0 \tag{41}$$

$$\frac{\Delta\bar{Y}}{\Delta\delta} = -\frac{\beta\bar{Y}}{1-\beta(1-\delta)} < 0 \tag{42}$$

This shows no difference with Keynesian theory where both multipliers are negative and have the same value. Solving for equilibrium consumption,

$$C_t = \beta(Y_{t-1} - \gamma - \delta Y_{t-1}) = \beta(1-\delta)Y_{t-1} - \beta\gamma \tag{43}$$

$$Y_{t-1} = \frac{C_t + \beta\gamma}{\beta(1-\delta)} \tag{44}$$

Substituting for the respective terms in (31), gives a second-order difference equation in consumption.

$$C_{t+2} - \beta(1+\alpha)(1-\delta)C_{t+1} + \alpha\beta(1-\delta)C_t = \beta(1-\delta)G_o - \beta\gamma \tag{45}$$

For equilibrium aggregate consumption,

$$\overline{C} = \frac{\beta(1-\delta)G_o - \beta\gamma}{1-\beta(1-\delta)} \tag{46}$$

Finally, for taxation T,

$$I_t = \alpha\beta(Y_{t-1} - T_{t-1} - Y_{t-2} + T_{t-2}) \tag{47}$$

Where $Y_t = \dfrac{T_t - \gamma}{\delta}$ from (34)

$$I_t = \alpha\beta\left(\frac{T_{t-1} - \gamma}{\delta} - T_{t-1} - \frac{T_{t-2} - \gamma}{\delta} + T_{t-2}\right) \tag{48}$$

Substituting in (31) results in

$$T_{t+2} - \beta(1+\alpha)(1-\delta)T_{t+1} + \alpha\beta(1-\delta)T_t = (1-\beta)\gamma + \delta G_o \tag{49}$$

Thus, for equilibrium tax revenue we have

$$\overline{T} = \frac{(1-\beta)\gamma + \delta G_o}{1-\beta(1-\delta)} \tag{50}$$

The equilibrium values for national income, consumption and tax we get are equal to those under the ordinary Keynesian national income model with taxes included, that is,

$$\overline{Y} = \frac{G_o - \beta\gamma}{1-\beta(1-\delta)} \qquad \overline{C} = \frac{\beta(1-\delta)G_o - \beta\gamma}{1-\beta(1-\delta)} \qquad \overline{T} = \frac{(1-\beta)\gamma + \delta G_o}{1-\beta(1-\delta)} \tag{51}$$

Furthermore, we can easily check the identity $\overline{C} = \beta(\overline{Y} - \overline{T})$ under the expanded multiplier-accelerator model, where consumption is a share of disposable income, or

$$\bar{C}=\beta(\bar{Y}-\bar{T})=\beta\left(\frac{(G_o-\beta\gamma}{1-\beta(1-\delta)}-\frac{(1-\beta)\gamma+\delta G_o}{1-\beta(1-\delta)}\right)=\frac{\beta(1-\delta)G_o-\beta\gamma}{1-\beta(1-\delta)} \tag{52}$$

We notice that both consumption and tax revenue are positively related to government spending. In the expanded tax multiplier-accelerator model we can again extend the period of change in consumption so that the new model becomes

$$Y_t=C_t+I_t+G_o \tag{53}$$
$$C_t=\beta(Y_{t-1}-T_{t-1}) \qquad 0<\beta<1 \tag{54}$$
$$I_t=\alpha(C_{t-1}-C_{t-2}) \qquad \alpha>0 \tag{55}$$
$$T_t=\gamma+\delta Y_t \qquad \gamma>0 \quad 0<\delta<1 \tag{56}$$

Substituting into the accelerator equation (55), we get:

$$I_t=\alpha\beta(Y_{t-2}-T_{t-2}-Y_{t-3}+T_{t-3}) \tag{57}$$
$$I_t=\alpha\beta(Y_{t-2}-\gamma-\delta Y_{t-2}-Y_{t-3}+\gamma+\delta Y_{t-3}),\text{ and} \tag{58}$$
$$C_t=\beta(1-\delta)Y_{t-1}-\beta\gamma \tag{59}$$

gives the following equation for national income

$$Y_{t+3}-\beta(1-\delta)Y_{t+2}-\alpha\beta(1-\delta)Y_{t+1}+\alpha\beta(1-\delta)Y_t=G_o-\beta\gamma \tag{60}$$
$$\bar{Y}=\frac{G_o-\beta\gamma}{1-\beta(1-\delta)} \tag{61}$$

Substituting for consumption in (53) in the extended case where $Y_{t-1}=\dfrac{C_t+\beta\gamma}{\beta(1-\delta)}$,

$$C_{t+3}-\beta(1-\delta)C_{t+2}-\alpha\beta(1-\delta)C_{t+1}+\alpha\beta(1-\delta)C_t=\beta(1-\delta)G_o-\beta\gamma \tag{62}$$
$$\bar{C}=\frac{\beta(1-\delta)G_o-\beta\gamma}{1-\beta(1-\delta)} \tag{63}$$

Finally, for tax in the extended version,

$$I_t=\alpha\beta(Y_{t-2}-T_{t-2}-Y_{t-3}+T_{t-3}) \tag{64}$$

where $\quad Y_t=\dfrac{T_t-\gamma}{\delta}$ from (56)

$$I_t=\alpha\beta\left(\frac{T_{t-2}-\gamma}{\delta}-T_{t-2}-\frac{T_{t-3}-\gamma}{\delta}+T_{t-3}\right)=\frac{\alpha\beta(1-\delta)}{\delta}(T_{t-2}-T_{t-3}) \tag{65}$$

which upon substitution in (53) becomes

$$T_{t+3} - \beta(1-\delta)T_{t+2} - \alpha\beta(1-\delta)T_{t+1} + \alpha\beta(1-\delta)T_t = (1-\beta)\gamma + \delta G_o \tag{66}$$

Thus, for equilibrium tax revenue we have

$$\overline{T} = \frac{(1-\beta)\gamma + \delta G_o}{1-\beta(1-\delta)} \tag{67}$$

These three results for national income, endogenous consumption and tax are consistent with our previous findings, where moving consumption backward by a period does not modify the multiplier-accelerator relationship. In the absence of tax when $\gamma, \delta = 0$ the standard values of the aggregate economic variables obtain, that is, $\overline{Y} = \frac{G_o}{1-\beta}$ and $\overline{C} = \frac{\beta G_o}{1-\beta}$, as in (19) and (21). The accelerator α is also missing in all equilibrium solutions, as in all previous versions of the model, having no effect on optimal values which originate from Keynesian theory.

4. An expanded multiplier-accelerator model with foreign trade included

An expanded version of the multiplier-accelerator model could involve the volume of international trade so that to see the effect of both the multiplier and accelerator on an open, rather than a closed, economy. In the standard Keynesian model of national income augmented by the volume of net exports the effect of exogenous exports on national income is positive. The model assumes exports to be exogenous and allows solving for the exports multiplier.

$$Y_t = C_t + I_t + G_o + X_o - M \tag{68}$$

$$C_t = \beta Y_{t-1} \qquad 0 < \beta < 1 \tag{69}$$

$$I_t = \alpha(C_t - C_{t-1}) \qquad \alpha > 0 \tag{70}$$

$$M_t = mY_t \qquad 0 < m < 1 \tag{71}$$

Solving for equilibrium national income under the new assumptions,

$$Y_{t+2} - \frac{\beta(1+\alpha)}{1+m}Y_{t+1} + \frac{\alpha\beta}{1+m}Y_t = \frac{G_o + X_o}{1+m} \tag{72}$$

At the steady state we have $\overline{Y} = Y_t = Y_{t+1} = Y_{t+2}$ so for equilibrium national income $\overline{Y}$ we have

$$\overline{Y} = \frac{G_o + X_o}{\left[1 - \frac{\beta(1+\alpha)}{1+m} + \frac{\alpha\beta}{1+m}\right](1+m)} = \frac{G_o + X_o}{1+m-\beta} \tag{73}$$

The equilibrium national income again lacks the accelerator coefficient and gives the value of the exports multiplier as

$$\frac{d\overline{Y}}{dX_o} = \frac{1}{1+m-\beta} \tag{74}$$

Expressing consumption,

$$\frac{C_{t+1}}{\beta} = C_t + \alpha C_t - \alpha C_{t-1} + G_o + X_o - \frac{m}{\beta} C_t \tag{75}$$

$$C_{t+2} - (\beta + \alpha\beta - m)C_{t+1} + \alpha\beta C_t = \beta(G_o + X_o) \tag{76}$$

For equilibrium aggregate consumption,

$$\overline{C} = \frac{\beta(G_o + X_o)}{1-\beta-\alpha\beta+m+\alpha\beta} = \frac{\beta(G_o + X_o)}{1-\beta+m} \tag{77}$$

For endogenous imports where $Y_t = \frac{M_t}{m}$ and $C_t = \beta\frac{M_{t-1}}{m}$ upon substitution in national income,

$$\frac{M_t}{m} = \frac{\beta}{m} M_{t-1} + \frac{\alpha\beta}{m}(M_{t-1} - M_{t-2}) + G_o + X_o - M_t \tag{78}$$

$$M_{t+2} - \frac{\beta(1+\alpha)}{1+m} M_{t+1} + \frac{\alpha\beta}{1+m} M_t = \frac{m}{1+m}(G_o + X_o) \tag{79}$$

For equilibrium imports $\overline{M}$ we obtain

$$\overline{M} = \frac{m(G_o + X_o)}{1+m-\beta} \tag{80}$$

which is consistent with $\overline{M} = m\overline{Y}$ in equilibrium. Augmenting the model by tax and foreign trade simultaneously, we solve

$$Y_t = C_t + I_t + G_o + X_o - M \tag{81}$$

$$C_t = \beta Y_{t-1} \qquad 0 < \beta < 1 \tag{82}$$

$$I_t = \alpha(C_t - C_{t-1}) \qquad \alpha > 0 \tag{83}$$

$$T_t = \gamma + \delta Y_t \qquad \gamma > 0 \quad 0 < \delta < 1 \tag{84}$$

$$M_t = mY_t \qquad 0 < m < 1 \tag{85}$$

Expressing aggregate consumption and aggregate investment,

$$C_t = \beta(1-\delta)Y_{t-1} - \beta\gamma \tag{86}$$

$$I_t = \alpha\beta(1-\delta)Y_{t-1} - \alpha\beta(1-\delta)Y_{t-2}, \tag{87}$$

and substituting in the national income equation (81),

$$Y_{t+2}-\frac{\beta(1-\delta)(1+\alpha)}{1+m}Y_{t+1}+\frac{\alpha\beta(1-\delta)}{1+m}Y_t=\frac{G_o+X_o-\beta\gamma}{1+m} \tag{88}$$

It produces the following equilibrium value for national income

$$\overline{Y}=\frac{G_o+X_o-\beta\gamma}{1+m-\beta(1-\delta)-\alpha\beta(1-\delta)+\alpha\beta(1-\delta)}=\frac{G_o+X_o-\beta\gamma}{1+m-\beta(1-\delta)} \tag{89}$$

which produces the exports multiplier

$$\frac{d\overline{Y}}{dX_o}=\frac{1}{1+m-\beta(1-\delta)} \tag{90}$$

and the government expenditure multiplier with the same value $\frac{d\overline{Y}}{dG_o}=\frac{1}{1+m-\beta(1-\delta)}$. These results are consistent with the previous finding about the exports multiplier in the absence of taxation, i.e., $\frac{d\overline{Y}}{dX_o}=\frac{1}{1+m-\beta}$. Similarly, in the absence of trade where the marginal propensity to import is assumed to be $m=0$ the government expenditure multiplier obtains the already known value $\frac{d\overline{Y}}{dG_o}=\frac{1}{1-\beta(1-\delta)}$. The combined effect is one where a higher marginal propensity to import m reduces the value of both multipliers and that of income tax δ is identical. We see that a higher income tax reduces both multipliers, whereas the effect of both income and non-income tax is negative on equilibrium national income.

5. Conclusions

Samuelson studied the interplay between the multiplier and accelerator, adding the accelerator coefficient to the standard Keynesian multiplier model. It could be expected that the accelerator effect might prevail over that of the multiplier and thatnational income drives aggregate investment, rather than the other way around. Using discrete time analysis and difference equations and solving the model in various contexts we do not find strict confirmation that the accelerator effect refutes or prevails over the multiplier effect. Just the opposite, we find that at the steady state, all equilibrium results of the Keynesian national-income model hold. The values of equilibrium national income are identical to those under Keynesian assumptions in the case of the simple national-income model and multiplier, the tax-augmented national-income model as well as the national-income model in the conditions of an open economy, that is, with foreign trade added.We solve the simple multiplier-accelerator model both

in present terms and under a prolonged period of acceleration, i.e., by extendingconsumption one period backward.Using second-order and third-order difference equations, we solve for equilibrium consumption, tax, and imports. All results confirm the validity of Keynesian theory where investment, government spending and exports have a favorable multiplying effect on national income through their multipliers. We derive two more multipliers, the income and the non-income tax multipliers which confirm Keynesian analysis, both being negative and taking the values previously derived by Keynes.

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