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Equivalence of Cournot and Bertrand Equilibria in Duopoly under Relative Profit Maximization: A General Analysis

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Abstract. We study the relationship between Cournot equilibrium and Bertrand equilibrium in duopoly with differentiated goods in which each firm maximizes its relative profit. We show that Cournot equilibrium and Bertrand equilibrium coincide under relative profit maximization even with general demand and cost functions. This result is due to the fact that a game of relative profit maximization in duopoly is a two-person zero-sum game.
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1. Introduction

We study the relationship between Cournot equilibrium and Bertrand equilibrium in duopoly with differentiated (substitutable or complementary) goods in which each firm maximizes its relative profit instead of its absolute profit. We consider general demand and cost functions. The cost functions of firms may be different each other, and the demand functions may be asymmetric. Mainly we show the following result.

Cournot equilibrium and Bertrand equilibrium coincide under relative profit maximization.

In recent years, maximizing relative profit instead of absolute profit by firms has aroused the interest of economists. See, for example, Gibbons & Murphy (1990); Lu (2011), Matsumura, Matsushima & Cato, (2013); (Satoh & Tanaka (2013) and Hattori & Tanaka (2014).

Theoretical justification of relative profit maximization is mainly based on evolutionary game theoretic point of view. Schaffer, (1989) demonstrates with a Darwinian model of economic natural selection that if firms have market power, profit-maximizers are not necessarily the best survivors. A unilateral deviation from Cournot equilibrium decreases the profit of the deviator, but decreases the other firm's profit even more. On the condition of being better than other competitors, firms that deviate from Cournot equilibrium achieve higher payoffs than the payoffs they receive under Cournot equilibrium. He defines the finite

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population evolutionarily stable strategy (FPESS), and shows that it is a strategy of a player that maximizes his relative payoff. This is according to the following fact.

If there are both absolute payoff maximizing players and relative payoff maximizing players, then the latter players earn more *absolute* payoffs than the former players. Thus, relative payoff maximizing strategy is more survival than absolute payoff maximizing strategy.

In Vega-Redondo (1997) it was shown that FPESS is a strategy that survives in the long run equilibrium or stochastically stable state of a dynamic stochastically evolutionary game developed by Kandori, Mailath & Rob (1993) and Rhode & Stegeman (1996). Referring to Alchian (1950) and Friedman (1953) he argued that it is relative rather than absolute performance which should in the end prove decisive in the long run. Also he showed that in a homogeneous good case if firms maximize relative profits, a Walrasian equilibrium can be induced, In the case of differentiated goods, however, the result under relative profit maximization is different from the Walrasian equilibrium.

Miller & Pazgal (2001) has shown the equivalence of price strategy and quantity strategy in a delegation game, in which owners of firms control managers of their firms seek to maximize an appropriate combination of absolute and relative profits, in a case of linear demand functions and constant marginal costs[1]. But in their analyses owners of firms still seek to maximize the absolute profits of their firms. On the other hand, in this paper we do not consider a delegation problem, and firms, or owners of firms, themselves seek to maximize their relative profits. In the Appendix we point out that if demand functions are linear, the equivalence of Cournot and Bertrand in a delegation problem with the weight on relative profit as a control variable holds; however, if demand functions are not linear, the equivalence is unlikely to hold.

A game of relative profit maximization in duopoly is a two-person zero-sum game. In Section 6 we present interpretations of our results from the point of view of zero-sum game theory.

In Tanaka (2013) a similar result was shown in a simple case where demand functions are symmetric and linear, firms have the same constant marginal costs. This paper extends this result to a case where demand functions are general and may be asymmetric, and two firms may have different cost functions and cost functions are general.

2. The model

There are two firms, A and B. They produce differentiated substitutable or complementary goods. The outputs of Firm A and B are denoted by x_A and x_B, the prices of their goods are denoted by p_A and p_B. The demand functions of the goods produced by the firms are

$$x_A = x_A(p_A, p_B),\ x_B = x_B(p_A, p_B).$$

Inverting them yields the following inverse demand functions

$$p_A = p_A(x_A, x_B),\ p_B = p_B(x_A, x_B).$$

[1] In their general model owners of firms have very strong control power over their managers beyond control of the objective function. Thus, that model has little connection with relative profit consideration.

From them the following relations are derived.

$$\frac{\partial x_A}{\partial p_A}=\frac{\frac{\partial p_B}{\partial x_B}}{\frac{\partial p_A}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_A}{\partial x_B}\frac{\partial p_B}{\partial x_A}},\ \frac{\partial x_B}{\partial p_B}=\frac{\frac{\partial p_A}{\partial x_A}}{\frac{\partial p_A}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_A}{\partial x_B}\frac{\partial p_B}{\partial x_A}},$$

$$\frac{\partial x_B}{\partial p_A}=-\frac{\frac{\partial p_B}{\partial x_A}}{\frac{\partial p_A}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_A}{\partial x_B}\frac{\partial p_B}{\partial x_A}},\ \frac{\partial x_A}{\partial p_B}=-\frac{\frac{\partial p_A}{\partial x_B}}{\frac{\partial p_A}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_A}{\partial x_B}\frac{\partial p_B}{\partial x_A}}.$$

We have

$$\frac{\partial x_A}{\partial p_A}<0,\ \frac{\partial x_B}{\partial p_B}<0,\ \frac{\partial p_A}{\partial x_A}<0,\ \frac{\partial p_B}{\partial x_B}<0.$$

If the goods of Firm A and B are substitutes,

$$\frac{\partial x_B}{\partial p_A}>0,\ \frac{\partial x_A}{\partial p_B}>0,\ \frac{\partial p_B}{\partial x_A}<0,\ \frac{\partial p_A}{\partial x_B}<0.$$

If they are complements,

$$\frac{\partial x_B}{\partial p_A}<0,\ \frac{\partial x_A}{\partial p_B}<0,\ \frac{\partial p_B}{\partial x_A}>0,\ \frac{\partial p_A}{\partial x_B}>0.$$

We assume that the effect of a change in the price (or output) of a good on its demand (or price) is larger than the effect on demand (or price) of the other good. This means

$$\left|\frac{\partial x_A}{\partial p_A}\right|>\left|\frac{\partial x_B}{\partial p_A}\right|,\ \left|\frac{\partial x_A}{\partial p_A}\right|>\left|\frac{\partial x_A}{\partial p_B}\right|,\ \left|\frac{\partial p_B}{\partial x_B}\right|>\left|\frac{\partial p_A}{\partial x_B}\right|,\ \left|\frac{\partial p_B}{\partial x_B}\right|>\left|\frac{\partial p_B}{\partial x_A}\right|,$$

and so on. Then,

$$\frac{\partial x_A}{\partial p_A}\frac{\partial x_B}{\partial p_B}-\frac{\partial x_A}{\partial p_B}\frac{\partial x_B}{\partial p_A}>0,\ \frac{\partial p_A}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_A}{\partial x_B}\frac{\partial p_B}{\partial x_A}>0 \tag{1}$$

hold. The absolute profits of Firm A and B in the Cournot model in which strategic variables of the firms are their outputs are written as

$$\pi_A(x_A,x_B)=p_A(x_A,x_B)x_A-c_A(x_A),$$

$$\pi_B(x_A,x_B)=p_B(x_A,x_B)x_B-c_B(x_B).$$

$c_A(\cdot)$ and $c_B(\cdot)$ are the cost functions of Firm A and B. We assume

$$p_A - c_A'(x_A) > 0,\ p_B - c_B'(x_B) > 0$$

at the Cournot and Bertrand equilibria under absolute and relative profit maximization, where $c_A'(x_A)$ and $c_B'(x_B)$ are the marginal cost functions of Firm A and B.

If the goods of the firms are substitutes,

$$\frac{\partial \pi_A}{\partial x_B} = \frac{\partial p_A}{\partial x_B} x_A < 0,\ \frac{\partial \pi_B}{\partial x_A} = \frac{\partial p_B}{\partial x_A} x_B < 0,$$

and if the goods are complements,

$$\frac{\partial \pi_A}{\partial x_B} = \frac{\partial p_A}{\partial x_B} x_A > 0,\ \frac{\partial \pi_B}{\partial x_A} = \frac{\partial p_B}{\partial x_A} x_B > 0.$$

On the other hand, the absolute profits of Firm A and B in the Bertrand model in which strategic variables of the firms are the prices of their goods are written as

$$\pi_A(x_A(p_A, p_B), x_B(p_A, p_B)) = p_A x_A(p_A, p_B) - c_A(x_A(p_A, p_B)),$$

$$\pi_B(x_A(p_A, p_B), x_B(p_A, p_B)) = p_B x_B(p_A, p_B) - c_B(x_B(p_A, p_B)),$$

If the goods of the firms are substitutes,

$$\frac{\partial \pi_A}{\partial p_B} = (p_A - c'_A)\frac{\partial x_A}{\partial p_B} > 0,\ \frac{\partial \pi_B}{\partial p_A} = (p_B - c'_B)\frac{\partial x_B}{\partial p_A} > 0,$$

and if the goods are complements,

$$\frac{\partial \pi_A}{\partial p_B} = (p_A - c'_A)\frac{\partial x_A}{\partial p_B} < 0,\ \frac{\partial \pi_B}{\partial p_A} = (p_B - c'_B)\frac{\partial x_B}{\partial p_A} < 0.$$

3. Cournot equilibrium under relative profit maximization

In this section and the next section we consider Cournot and Bertrand equilibria in duopoly under relative profit maximization. We define the relative profit of each firm as the difference between its absolute profit and the absolute profit of the rival firm.

Denote the relative profit of Firm A by Π_A and that of Firm B by Π_B. Then, we have

$$\Pi_A = \pi_A - \pi_B,\ \Pi_B = \pi_B - \pi_A = -\Pi_A. \quad (2)$$

The first order conditions for relative profit maximization of Firm A and B, respectively, with respect to x_A and x_B are

$$\frac{\partial \Pi_A}{\partial x_A}=\frac{\partial \pi_A}{\partial x_A}-\frac{\partial \pi_B}{\partial x_A}=0,\ \frac{\partial \Pi_B}{\partial x_B}=\frac{\partial \pi_B}{\partial x_B}-\frac{\partial \pi_A}{\partial x_B}=0, \tag{3}$$

where

$$\frac{\partial \pi_A}{\partial x_A}=p_A-c_A'+\frac{\partial p_A}{\partial x_A}x_A,\ \frac{\partial \pi_B}{\partial x_B}=p_B-c_B'+\frac{\partial p_B}{\partial x_B}x_B.$$

Since $\Pi_B=-\Pi_A$ we have

$$\frac{\partial \Pi_B}{\partial x_A}=-\frac{\partial \Pi_A}{\partial x_A},\ \frac{\partial \Pi_A}{\partial x_B}=-\frac{\partial \Pi_B}{\partial x_B}.$$

We assume the existence of an interior Cournot equilibrium.

4. Bertrand equilibrium under relative profit maximization

In the Bertrand model the first order conditions for relative profit maximization of Firm A and B, respectively, with respect to p_A and p_B are

$$\frac{\partial \Pi_A}{\partial p_A}=\frac{\partial \pi_A}{\partial p_A}-\frac{\partial \pi_B}{\partial p_A}=0,\ \frac{\partial \Pi_B}{\partial p_B}=\frac{\partial \pi_B}{\partial p_B}-\frac{\partial \pi_A}{\partial p_B}=0. \tag{4}$$

We can show

$$\frac{\partial \pi_A}{\partial x_A}\frac{\partial x_A}{\partial p_A}+\frac{\partial \pi_A}{\partial x_B}\frac{\partial x_B}{\partial p_A}=(p_A-c_A')\frac{\partial x_A}{\partial p_A}+\left(\frac{\partial p_A}{\partial x_A}\frac{\partial x_A}{\partial p_A}+\frac{\partial p_A}{\partial x_B}\frac{\partial x_B}{\partial p_A}\right)x_A$$

$$=(p_A-c_A')\frac{\partial x_A}{\partial p_A}+\left(\frac{\frac{\partial p_A}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_A}{\partial x_B}\frac{\partial p_B}{\partial x_A}}{\frac{\partial p_A}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_A}{\partial x_B}\frac{\partial p_B}{\partial x_A}}\right)x_A=(p_A-c_A')\frac{\partial x_A}{\partial p_A}+x_A=\frac{\partial \pi_A}{\partial p_A},$$

$$\frac{\partial \pi_B}{\partial x_A}\frac{\partial x_A}{\partial p_A}+\frac{\partial \pi_B}{\partial x_B}\frac{\partial x_B}{\partial p_A}=\left(\frac{\partial p_B}{\partial x_A}\frac{\partial x_A}{\partial p_A}+\frac{\partial p_B}{\partial x_B}\frac{\partial x_B}{\partial p_A}\right)x_B+(p_B-c_B')\frac{\partial x_B}{\partial p_A}$$

$$=\left(\frac{\frac{\partial p_B}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_B}{\partial x_A}\frac{\partial p_B}{\partial x_B}}{\frac{\partial p_A}{\partial x_A}\frac{\partial p_B}{\partial x_B}-\frac{\partial p_A}{\partial x_B}\frac{\partial p_B}{\partial x_A}}\right)x_B+(p_B-c_B')\frac{\partial x_B}{\partial p_A}=(p_B-c_B')\frac{\partial x_B}{\partial p_A}=\frac{\partial \pi_B}{\partial p_A}.$$

Similarly,

$$\frac{\partial \pi_B}{\partial p_B} = \frac{\partial \pi_B}{\partial x_A}\frac{\partial x_A}{\partial p_B} + \frac{\partial \pi_B}{\partial x_B}\frac{\partial x_B}{\partial p_B}, \frac{\partial \pi_A}{\partial p_B} = \frac{\partial \pi_A}{\partial x_A}\frac{\partial x_A}{\partial p_B} + \frac{\partial \pi_A}{\partial x_B}\frac{\partial x_B}{\partial p_B}.$$

are obtained. Therefore, (4) is rewritten as

$$\frac{\partial \Pi_A}{\partial p_A} = \left(\frac{\partial \pi_A}{\partial x_A} - \frac{\partial \pi_B}{\partial x_A}\right)\frac{\partial x_A}{\partial p_A} + \left(\frac{\partial \pi_A}{\partial x_B} - \frac{\partial \pi_B}{\partial x_B}\right)\frac{\partial x_B}{\partial p_A} = \frac{\partial \Pi_A}{\partial x_A}\frac{\partial x_A}{\partial p_A} - \frac{\partial \Pi_B}{\partial x_B}\frac{\partial x_B}{\partial p_A} = 0,$$

$$\frac{\partial \Pi_B}{\partial p_B} = \left(\frac{\partial \pi_B}{\partial x_B} - \frac{\partial \pi_A}{\partial x_B}\right)\frac{\partial x_B}{\partial p_B} + \left(\frac{\partial \pi_B}{\partial x_A} - \frac{\partial \pi_A}{\partial x_A}\right)\frac{\partial x_A}{\partial p_B} = \frac{\partial \Pi_B}{\partial x_B}\frac{\partial x_B}{\partial p_B} - \frac{\partial \Pi_A}{\partial x_A}\frac{\partial x_A}{\partial p_B} = 0.$$

By (1) these equations mean

$$\frac{\partial \Pi_A}{\partial x_A} = 0, \frac{\partial \Pi_B}{\partial x_B} = 0. \tag{5}$$

(5) is the same as (3). Therefore, the first order conditions for relative profit maximization in the Cournot model are the same as those in the Bertrand model.

5. Interior Cournot and Bertrand equilibria

Let x_A^* and x_B^* be the outputs, p_A^* and p_B^* be the prices at the Cournot and Bertrand equilibria derived from the first order conditions. Suppose $x_A < x_A^*$ and $x_B < x_B^*$. Then, $p_A > p_A^*$ and $p_B > p_B^*$. Since (x_A^*, x_B^*) is the interior Cournot equilibrium, $\frac{\partial \Pi_A}{\partial x_A} > 0$ and $\frac{\partial \Pi_B}{\partial x_B} > 0$. Since $\frac{\partial x_A}{\partial p_A} < 0$, $\frac{\partial x_B}{\partial p_B} < 0$, $|\frac{\partial x_A}{\partial p_A}| > |\frac{\partial x_B}{\partial p_A}|$ and $|\frac{\partial x_B}{\partial p_B}| > |\frac{\partial x_A}{\partial p_B}|$, we have

$$\frac{\partial \Pi_A}{\partial p_A} < 0, \frac{\partial \Pi_B}{\partial p_B} < 0.$$

Suppose $x_A > x_A^*$ and $x_B > x_B^*$. Then, $p_A < p_A^*$ and $p_B < p_B^*$. Since (x_A^*, x_B^*) is the interior Cournot equilibrium, $\frac{\partial \Pi_A}{\partial x_A} < 0$ and $\frac{\partial \Pi_B}{\partial x_B} < 0$. Then, we have

$$\frac{\partial \Pi_A}{\partial p_A} > 0, \frac{\partial \Pi_B}{\partial p_B} > 0.$$

These results mean that the Bertrand equilibrium derived from the first order condition is an interior equilibrium. Then, we get the following proposition.

Proposition 1 *Under relative profit maximization in duopoly with differentiated goods the Cournot equilibrium and the Bertrand equilibrium are equivalent; and an interior Bertrand equilibrium exists if and only if an interior Cournot equilibrium exists.*

6. Zero-sum game interpretation of the equivalence of Cournot and Bertrand equilibria

As expressed in (2) a game of relative profit maximization in duopoly is a two-person zero-sum game. Generally the conditions for the Bertrand equilibrium are as follows.

$$\frac{\partial \Pi_A}{\partial p_A} = \frac{\partial \Pi_A}{\partial x_A}\frac{\partial x_A}{\partial p_A} + \frac{\partial \Pi_A}{\partial x_B}\frac{\partial x_B}{\partial p_A} = 0, \tag{6}$$

$$\frac{\partial \Pi_B}{\partial p_B} = \frac{\partial \Pi_B}{\partial x_A}\frac{\partial x_A}{\partial p_B} + \frac{\partial \Pi_B}{\partial x_B}\frac{\partial x_B}{\partial p_B} = 0. \tag{7}$$

Since the game is zero-sum, we have

$$\frac{\partial \Pi_B}{\partial x_A} = -\frac{\partial \Pi_A}{\partial x_A}, \frac{\partial \Pi_A}{\partial x_B} = -\frac{\partial \Pi_B}{\partial x_B}.$$

Then, (6) and (7) are reduced to

$$\frac{\partial \Pi_A}{\partial x_A}\frac{\partial x_A}{\partial p_A} - \frac{\partial \Pi_B}{\partial x_B}\frac{\partial x_B}{\partial p_A} = 0, \frac{\partial \Pi_A}{\partial x_A}\frac{\partial x_A}{\partial p_B} - \frac{\partial \Pi_B}{\partial x_B}\frac{\partial x_B}{\partial p_B} = 0.$$

Since $\frac{\partial x_A}{\partial p_A}\frac{\partial x_B}{\partial p_B} - \frac{\partial x_A}{\partial p_B}\frac{\partial x_B}{\partial p_A} > 0$, we obtain $\frac{\partial \Pi_A}{\partial x_A} = \frac{\partial \Pi_B}{\partial x_B} = 0$.

On the other hand, under absolute profit maximization the conditions for the Bertrand equilibrium are

$$\frac{\partial \pi_A}{\partial p_A} = \frac{\partial \pi_A}{\partial x_A}\frac{\partial x_A}{\partial p_A} + \frac{\partial \pi_A}{\partial x_B}\frac{\partial x_B}{\partial p_A} = 0.$$

$$\frac{\partial \pi_B}{\partial p_B} = \frac{\partial \pi_B}{\partial x_B}\frac{\partial x_B}{\partial p_B} + \frac{\partial \pi_B}{\partial x_A}\frac{\partial x_A}{\partial p_B} = 0.$$

Since $\frac{\partial \pi_B}{\partial x_A} = \frac{\partial p_B}{\partial x_A}x_B \neq 0$ and $\frac{\partial \pi_A}{\partial x_B} = \frac{\partial p_A}{\partial x_B}x_A \neq 0$, $\frac{\partial \pi_A}{\partial x_A} = 0$ and $\frac{\partial \pi_B}{\partial x_B} = 0$ do not hold. Thus, the equivalence of Cournot and Bertrand equilibria under relative profit maximization is due to the zero-sum game property of a game of relative profit maximization.

7. Comparison with absolute profit maximization

At the Cournot equilibrium under absolute profit maximization the first order conditions for Firm A and B are

$$\frac{\partial \pi_A}{\partial x_A} = 0, \frac{\partial \pi_B}{\partial x_B} = 0.$$

When x_A and x_B satisfy these equations, the following relations are obtained.

1. If the goods of the firms are substitutes,

$$\frac{\partial \Pi_A}{\partial x_A} = -\frac{\partial \pi_B}{\partial x_A} = -\frac{\partial p_B}{\partial x_A} x_B > 0, \frac{\partial \Pi_B}{\partial x_B} = -\frac{\partial \pi_A}{\partial x_B} = -\frac{\partial p_A}{\partial x_B} x_A > 0.$$

2. If the goods of the firms are complements,

$$\frac{\partial \Pi_A}{\partial x_A} = -\frac{\partial \pi_B}{\partial x_A} = -\frac{\partial p_B}{\partial x_A} x_B < 0, \frac{\partial \Pi_B}{\partial x_B} = -\frac{\partial \pi_A}{\partial x_B} = -\frac{\partial p_A}{\partial x_B} x_A < 0.$$

Then, we get the following results.

Proposition 2

1. If the goods of the firms are substitutes, the equilibrium outputs at the Cournot equilibrium under relative profit maximization are larger than the equilibrium outputs at the Cournot equilibrium under absolute profit maximization.
2. If the goods of the firms are complements, the equilibrium outputs at the Cournot equilibrium under relative profit maximization are smaller than the equilibrium outputs at the Cournot equilibrium under absolute profit maximization.

In the Bertrand model the conditions of absolute profit maximization for Firm A and B are

$$\frac{\partial \pi_A}{\partial p_A} = 0, \frac{\partial \pi_B}{\partial p_B} = 0.$$

When p_A and p_B satisfy these equations, the following relations are obtained.

1. If the goods of the firms are substitutes,

$$\frac{\partial \Pi_A}{\partial p_A} = -\frac{\partial \pi_B}{\partial p_A} = -(p_B - c_B')\frac{\partial x_B}{\partial p_A} < 0, \frac{\partial \Pi_B}{\partial p_B} = -(p_A - c_A')\frac{\partial x_A}{\partial p_B} < 0.$$

2. If the goods of the firms are complements,

$$\frac{\partial \Pi_A}{\partial p_A} = -\frac{\partial \pi_B}{\partial p_A} = -(p_B - c_B')\frac{\partial x_B}{\partial p_A} > 0, \frac{\partial \Pi_B}{\partial p_B} = -(p_A - c_A')\frac{\partial x_A}{\partial p_B} > 0.$$

Then, we get the following results.

Proposition 3

1. If the goods of the firms are substitutes, the equilibrium prices at the Bertrand equilibrium under relative profit maximization are lower than the equilibrium prices at the Bertrand equilibrium under absolute profit maximization.
2. If the goods of the firms are complements, the equilibrium prices at the Bertrand equilibrium under relative profit maximization are higher than the equilibrium prices at the Bertrand equilibrium under absolute profit maximization.

Appendix

Note on the equivalence of Cournot and Bertrand equilibria in a delegation problem

Consider a duopoly in which Firm A and B produce differentiated goods with no cost. Inverse and ordinary demand functions are

$$p_A = p_A(x_A, x_B),\ p_B = p_B(x_A, x_B),\ x_A = x_A(p_A, p_B),\ x_B = x_B(p_A, p_B).$$

p_A and p_B are the prices, and x_A and x_B are the outputs of, respectively, Firm A and Firm B. We assume that inverse and ordinary demand functions are symmetric. Then, at the Cournot and Bertrand equilibria we have $x_A = x_B$, $p_A = p_B$, $\frac{\partial p_A}{\partial x_A} = \frac{\partial p_B}{\partial x_B}$, $\frac{\partial p_A}{\partial x_B} = \frac{\partial p_B}{\partial x_A}$, $\frac{\partial x_A}{\partial p_A} = \frac{\partial x_B}{\partial p_B}$, $\frac{\partial x_A}{\partial p_B} = \frac{\partial x_B}{\partial p_A}$, and so on. At the symmetric equilibria we get

$$\frac{\partial x_A}{\partial p_A} = \frac{\frac{\partial p_A}{\partial x_A}}{\left(\frac{\partial p_A}{\partial x_A} - \frac{\partial p_A}{\partial x_B}\right)\left(\frac{\partial p_A}{\partial x_A} + \frac{\partial p_A}{\partial x_B}\right)},\ \frac{\partial x_A}{\partial p_B} = -\frac{\frac{\partial p_A}{\partial x_B}}{\left(\frac{\partial p_A}{\partial x_A} - \frac{\partial p_A}{\partial x_B}\right)\left(\frac{\partial p_A}{\partial x_A} + \frac{\partial p_A}{\partial x_B}\right)}. \tag{8}$$

The objective functions of the managers of Firm A and B in the Cournot case are

$$\phi_A = p_A(x_A, x_B)x_A - \theta_A p_B(x_A, x_B)x_B,\ \phi_B = p_A(x_A, x_B)x_A - \theta_B p_A(x_A, x_B)x_A,$$

and those in the Bertrand case are

$$\psi_A = p_A x_A(p_A, p_B) - \sigma_B p_B x_B(p_A, p_B),\ \psi_B = p_B x_B(p_A, p_B) - \sigma_B p_A x_A(p_A, p_B),$$

where θ_A, θ_B, σ_A and σ_B are constants. ϕ_A, ϕ_B, ψ_A and ψ_B are the weighted sums of absolute and relative profits. The managers of the firms determine, respectively, x_A and x_B to maximize, respectively, ϕ_A and ϕ_B in the Cournot case, and determine, respectively, p_A and p_B to maximize, respectively, ψ_A and ψ_B in the Bertrand case. The owner of each firm determines the value of θ_A or θ_B in the Cournot case, σ_A or σ_B in the Bertrand case to maximize the absolute profit, $p_A x_A$ or $p_B x_B$.

By straightforward calculations we find that the equilibrium values of θ_A, θ_B, σ_A and σ_B at the symmetric equilibria satisfy the following equations:

$$\theta_B = \theta_A = \frac{(1-\theta_A)\left(\frac{\partial p_A}{\partial x_B} + x_A \frac{\partial^2 p_A}{\partial x_A x_B}\right)}{2\frac{\partial p_A}{\partial x_A} + x_A \frac{\partial^2 p_A}{\partial x_A^2} - \theta_A x_A \frac{\partial^2 p_A}{\partial x_B^2}},\ \sigma_B = \sigma_A = \frac{(1-\sigma_A)\left(\frac{\partial x_A}{\partial p_B} + p_A \frac{\partial^2 x_A}{\partial p_A p_B}\right)}{2\frac{\partial x_A}{\partial p_A} + p_A \frac{\partial^2 x_A}{\partial p_A^2} - \sigma_A p_A \frac{\partial^2 x_A}{\partial p_B^2}}. \tag{9}$$

They are complicated. However, if demand functions are linear, we obtain

$$\theta_A = \frac{\frac{\partial p_A}{\partial x_B}}{2\frac{\partial p_A}{\partial x_A} + \frac{\partial p_A}{\partial x_B}},\ \sigma_A = \frac{\frac{\partial x_A}{\partial p_B}}{2\frac{\partial x_A}{\partial p_A} + \frac{\partial x_A}{\partial p_B}}.$$

The first order condition for the firms at the symmetric Cournot equilibrium is written as

$$p_A + x_A\frac{\partial p_A}{\partial x_A} - \theta_A x_A\frac{\partial p_A}{\partial x_B} = 0. \tag{10}$$

The first order condition for the firms at the symmetric Bertrand equilibrium is written as

$$x_A + p_A\frac{\partial x_A}{\partial p_A} - \sigma_A p_B\frac{\partial x_B}{\partial p_A} = 0. \tag{11}$$

They hold in both linear and non-linear cases. In a non-linear case the equivalence of Cournot and Bertrand equilibria is unlikely to hold with (9).

However, in a linear case (10) and (11) are rewritten as

$$p_A + x_A\frac{\partial p_A}{\partial x_A} - \frac{\frac{\partial p_A}{\partial x_B}}{2\frac{\partial p_A}{\partial x_A}+\frac{\partial p_A}{\partial x_B}}x_A\frac{\partial p_A}{\partial x_B} = p_A + \frac{\left(2\frac{\partial p_A}{\partial x_A}-\frac{\partial p_A}{\partial x_B}\right)\left(\frac{\partial p_A}{\partial x_A}+\frac{\partial p_A}{\partial x_B}\right)}{2\frac{\partial p_A}{\partial x_A}+\frac{\partial p_A}{\partial x_B}}x_A = 0, \tag{12}$$

$$x_A + p_A\frac{\partial x_A}{\partial p_A} - \frac{\frac{\partial x_A}{\partial p_B}}{2\frac{\partial x_A}{\partial p_A}+\frac{\partial x_A}{\partial p_B}}p_A\frac{\partial x_A}{\partial p_B} = x_A + \frac{\left(2\frac{\partial x_A}{\partial p_A}-\frac{\partial x_A}{\partial p_B}\right)\left(\frac{\partial x_A}{\partial p_A}+\frac{\partial x_A}{\partial p_B}\right)}{2\frac{\partial x_A}{\partial p_A}+\frac{\partial x_A}{\partial p_B}}p_A = 0. \tag{13}$$

From (8) we have

$$2\frac{\partial x_A}{\partial p_A}+\frac{\partial x_A}{\partial p_B} = \frac{2\frac{\partial p_A}{\partial x_A}-\frac{\partial p_A}{\partial x_B}}{\left(\frac{\partial p_A}{\partial x_A}-\frac{\partial p_A}{\partial x_B}\right)\left(\frac{\partial p_A}{\partial x_A}+\frac{\partial p_A}{\partial x_B}\right)},\ 2\frac{\partial x_A}{\partial p_A}-\frac{\partial x_A}{\partial p_B} = \frac{2\frac{\partial p_A}{\partial x_A}+\frac{\partial p_A}{\partial x_B}}{\left(\frac{\partial p_A}{\partial x_A}-\frac{\partial p_A}{\partial x_B}\right)\left(\frac{\partial p_A}{\partial x_A}+\frac{\partial p_A}{\partial x_B}\right)},$$

$$\frac{\partial x_A}{\partial p_A}+\frac{\partial x_A}{\partial p_B} = \frac{1}{\frac{\partial p_A}{\partial x_A}+\frac{\partial p_A}{\partial x_B}}.$$

Then, (12) and (13) are the same equations. Therefore, if demand functions are linear, the Cournot and Bertrand equilibria are equivalent, however if demand functions are not linear, the equivalence is unlikely to hold.

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