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The Law of Growth

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Abstract. A simple law of growth is formulated. It links growth trajectories with the driving force and thus furnished an easy way to study and interpret the mechanisms of growth. The application of this law is illustrated by examples.
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1. Introduction

The aim of this publication is to formulate a simple law of growth, which could be used to study, interpret and understand any type of growth including economic growth described by the Gross Domestic Product or by income per capita. This comment will make it clear why this discussion is applicable to the study of economic growth. This law will link directly the trajectory of growth with the driving force. The aim here is to facility an easy and transparent way for studying the mechanism of growth because the mechanism of growth is defined by the associated driving force.

If we can link the driving force with the growth trajectory we can then easily check our interpretation of the mechanism of growth. We can use various types of forces to test whether proposed mechanism is in agreement with the empirical evidence. We can extend our study to predict growth assuming that the mechanism of growth is going to be unchanged, but we can also predict growth by assuming a different mechanism of growth. In such a case, we can also use the general law of growth but with a new, suitably defined driving force.

2. The law of growth

2.1. The definition

The well-known principle in scientific investigations is: *Entia non sunt multiplicanda praeter necessitatem.* Before looking for complicated explanations or formulations, it is always advisable to adopt the simplest possible approach. Complicated explanations might be impressive and in some cases even unavoidable but the simplest solutions are always more attractive.

It is well-known and generally accepted that any growth can, and usually is, described by the *growth rate*. Once we know the growth rate we can immediately understand whether the growth is fast or slow. We can even have a certain degree of understanding of its possible future, whether it can be sustained or not, whether

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it is too slow and should be speeded up, if possible, or maybe that it is too fast and should be slowed down.

Thus, for instance, economic growth is routinely described using the *percentage* of the annual increase or decline. The growth of human population is also characterised in the same way.

Often, in order to understand growth, the growth rate is converted into the *doubling time*. The equivalent quantity for radioactive isotopes is the half-lifetime calculated using the decay rate. If we have a radioactive material, which decays within seconds, we do not have to worry too much about its harmful effects. However, if we have radioactive contamination containing substantial amounts of radioactive isotopes with half-lifetime of millions of billions of years we can be sure that we have a serious problem. Likewise, if the doubling time for a spread of certain infectious diseases is millions of years we do not need to be worried but if the doubling time is measured in days, then again we have a serious problem. We do not have to carry out any laborious calculation. A simple calculation assuming a constant doubling time or a constant growth rate can lead easily to an approximately correct answer, which in many cased is quite acceptable.

Even though the simple formula for calculating the doubling time by dividing 70 or 69.3 by the growth rate expressed in per cent is applicable only to the exponential growth, and should never be applied to any other type of growth, such calculations are still carried out for other types of growth because we know that if we calculate the growth rate or the corresponding doubling time we can have a little better understanding of a given process.

It seems to be obvious that the growth rate reflects the mechanism of growt hand that there must be a close connection between the growth rate and the driving force of growth. The *simplest* way of describing this close connection is to assume that the growth rate is directly proportional to the driving force:

$$G = \kappa F, \tag{1}$$

where G is the growth rate, F is the driving force, and κ is a constant, which we could call *the growth promoting factor*, or *the compliance*, because the larger is the parameter κ, the faster is the growth.

Growth rate is defined as:

$$G(t) \equiv \frac{1}{S(t)} \frac{dS(t)}{dt}, \tag{2}$$

where $S(t)$ is the size of the growing entity, and t is time.

This quantity is sometimes labelled unnecessarily and confusingly as *the relative growth rate* to distinguish it from another redundant and confusing term *the absolute growth rate*, which describes just the change in the size of the growing entity per unit of time, i.e. dS/dt. To make it even more confusing, the term *absolute growth rate* is sometimes replaced by *the growth rate* (e.g. Karev & Kareva, 2014) or by *the exponential growth rate*.

This needless confusion could be easily avoided by leaving the well-known *growth rate*, as defined by the eqn (2), alone. It is a widely-used quantity applicable not only to the exponential growth but also to any other type of growth. All descriptions of growth in terms of per cent of the increase or in terms of the doubling time use the growth rate defined by the eqn (2), so the use of the term: *the absolute growth rate*, for this well-known growth rate represents an unnecessary and confusing aberration. We should always use the term *growth rate* only for the quantity defined by the eqn (2).

If we insist on using dS/dt to describe growth, we should never create confusion by associating it with the term *growth rate* but we should simply call it the *absolute change*. We do not create science by introducing complicated and confusing terms.

The eqn (1) represents the simplest, general *law of growth*. There could be many other ways of linking the growth rate with the driving force but we have assumed the simplest relation. We call the eqn (1) the law of growth rather than the model of growth because this equation can be used to formulate a variety of models of growth, some of them already well known, but many of them yet unknown. Rather than using the existing models, such as exponential or logistic, even if their application could be questionable, we can *tailor* the models of growth to the studied processes and by doing so we can then try to explain the mechanisms of growth described by the relevant driving force.

The eqn (1) can be rewritten as

$$F = rG \, , \tag{3}$$

where $r = \kappa^{-1}$. This is another representation of the general law of growth.

2.2. Analogies

In the form given by the eqn (3), the law of growth is similar to the Newton's second law of motion:

$$F = ma \, , \tag{4}$$

where *m* is the mass of a physical object and a is the acceleration.

Newton's law describes the *dynamics of physical objects*. If the driving force is zero, the acceleration is zero, which means that the physical object is either stationary or that it moves along a straight line with a constant velocity.

The law of growth describes the *dynamics of growing entities*. If the driving force is zero, the growth rate is zero and the size of the growing entity remains constant.

In the Newton's law, *m* is the mass of the physical object. The larger is *m* the larger force has to be used to have the same acceleration. The equivalent parameter in the law of growth is *r*, which can be interpreted as the *resistance* to growth. The larger is *r* the larger must be the driving force to have the same intensity of growth.

Acceleration is a well-known quantity and because of it, Newton's law can be used easily to understand the dynamics of physical objects. Growth rate is also a well-known quantity and because of it, the general law of growth can be also used easily to understand the dynamics of growing entities.

In its explicit form, Newton's second law of motion can be expressed as

$$F(t) = m \frac{d^2 s(t)}{dt^2} \, , \tag{5}$$

where $s(t)$ is the trajectory of the moving object. The dynamics of the moving object is explained by linking the trajectory $s(t)$ with the driving force $F(t)$.

Likewise, in its explicit form, the law of growth can be expressed as

$$F(t) = r\frac{1}{S(t)}\frac{dS(t)}{dt}. \tag{6}$$

The dynamics of the growing entity (the mechanism of growth) is explained by linking the size $S(t)$ of the growing entity with the driving force $F(t)$.

For physical objects, the driving force, i.e. the mechanism of motion, is *reflected* in the acceleration $a(t)$ and in the corresponding trajectory $s(t)$. If the driving force (the mechanism) is known, we can use it to calculate the corresponding trajectory of a moving object. However, if the trajectory is known but the driving force (the mechanism) is unknown, we can assume a driving force (a mechanism) and calculate the corresponding trajectory $s(t)$. If our calculations agree with relevant data, we can then claim that we have *explained* the mechanism of the moving object.

For growing entities, the driving force, i.e. the mechanism of growth, is *reflected* in the growth rate $G(t)$ and in the corresponding trajectory describing the size $S(t)$ of the growing entity. If the driving force (the mechanism) is known, we can use it to calculate the corresponding trajectory of a growing entity. However, if the trajectory is known but the driving force (the mechanism of growth) is unknown, we can assume a driving force (a mechanism of growth) and calculate the corresponding trajectory $S(t)$. If our calculations agree with relevant data, we can then claim that we have *explained* the mechanism of growth.

We can calculate the trajectory $s(t)$ of a physical object directly from the acceleration without using the Newton's law. However, to understand why a moving object follows a certain trajectory we have to understand the driving force, and the link between the driving force and the trajectory is given conveniently by the Newton's law of motion.

Likewise, we can calculate the trajectory $S(t)$ of the growing entity directly from the growth rate without using the law of growth. However, to understand why the growth follows a certain trajectory we have to understand the driving force, and the link between the driving force and the trajectory is given by the law of growth.

The difference between the Newton's law of motion and the law of growth is that while Newton's law is a three-dimensional vector, the law of growth is a scalar, which makes the description of growth much simpler than the description of the dynamics of physical objects.

In Newton's law, mass m represents an *intrinsic* property of a physical object. For the law of growth, resistance to growth, r, might have a broader interpretation. It might represent an intrinsic property of a growing entity but it might also depend on exogenous conditions. In this respect, there is a close similarity between the law of growth and other similar simple and well-known laws listed in Table 1.

Table 1. *Summary of the analogous laws*

Name	Law	Explanation
Newton's law	$F = ma$	F – driving force; m – mass; a – acceleration
Ohm's law	$U = RI$	U – potential; R – resistance; I – current
Hagen–Poiseuille law	$\Delta P = rV$	ΔP – pressure difference; r – resistance; V – volume velocity
Darcy's law	$\nabla P = rV_F$	∇P – pressure gradient; r – resistance to flow; V_F – volumetric flux
Fourier's law	$\nabla T = rH$	∇T – temperature gradient; r – thermal resistivity; H – heat flux
Law of growth	$F = rG$	F – driving force; r – resistance to growth; G – growth rate

For instance, the law of growth is similar to Ohm's law, $U = RI$, describing the flow of electricity. The electrical potential, *U*, plays here the role of the driving force [*cf* eqn. (3)] and *R* is the resistance to flow. The parameter κ in the law of growth given by the eqn (1) plays similar role as the *conductance*, $1/R$, in the Ohm's law. Resistance, *R*, is determined by the intrinsic property of the conducting material (*electrical resistivity*) but it also depends on the geometrical dimensions of the conducting medium (its length and the cross-section area). Furthermore, while resistivity characterises an *intrinsic* property of the conducting medium, it also depends on the temperature.

The law of growth is also similar to the Hagen–Poiseuille law describing the flow of fluids through cylindrical conduits. In this law, pressure difference plays the role of the driving force. Resistance to flow depends not only on the intrinsic property of a given liquid (*viscosity*) but also on the geometrical dimensions of the cylindrical conduit (its length and its radius). However, viscosity depends also on the temperature.

The law of Hagen–Poiseuille is usually expressed using pressure difference and *volume velocity* but it can be also presented using *pressure gradient* and *volumetric flux* (volume velocity per unit area). In this from it resembles the Darcy's law describing the flow of fluids through porous medium where the resistance to flow is given by the ratio of *viscosity* and *permeability* both depending on the temperature.

The law of growth is also similar to the Fourier's law describing the conductive heat transfer, where *heat flux* (energy transferred per units of time and area) is given by the product of *conductivity* and the temperature gradient, in the same way as the growth rate is given by the product of κ and the driving force in the eqn (1). Temperature gradient plays the role of the driving force while thermal conductivity is equivalent to the parameter κ. The inverse value of thermal conductivity is *thermal resistivity*. This quantity characterises the *intrinsic* property of the heat transferring medium but it also depends on the temperature.

3. Examples of applications of the law of growth

We shall now give a few simple examples how the law of growth can be used in the study of the mechanism of growth. We shall show how we can tailor our interpretations of growth to understand better its mechanism. We do not have to be restricted to using just a certain, limited range of models of growth. We can design and use our own models. We can explore a wide range of mechanisms of growth and check, which of them gives the best description of data. In general, we might have to solve the relevant differential equations numerically but in many cases, we might have a convenient analytical solution.

3.1. Exponential growth

We might assume, for instance, that the driving force is constant,

$$F(t) = c\,. \tag{7}$$

It is the simplest force of growth. By being constant it, obviously, does not depend on time or on the size of the growing entity. This comment might sound trivial but it is important to understand that for other types of growth the driving force can depend not only on time but also on the size of the growing entity, or on the combination of time and size, and that all such options will describe the multitude of possible *models* of growth.

If we use this force in the eqn (1) we shall get

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = k, \tag{8}$$

where $k \equiv c/r = c\kappa$.

The solution of this differential equation is the exponential function,

$$S(t) = Ce^{kt} \tag{9}$$

where C is related to the constant of integration.

Now, we can understand this growth a little better because we know where it belongs. It belongs to a specific class or the type of growth, for which the driving force is constant. For the same intensity c of the driving force, the smaller is the resistance r or the larger is the compliance (or growth promoting factor) κ, the larger is the growth rate k and the faster is the exponential growth.

3.2. The extension of the exponential growth

Let us now use a more general example when the driving force is not constant but depends on time,

$$F(t) = f(t). \tag{10}$$

If we use this force in the eqn (1) we shall have

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = \kappa f(t). \tag{11}$$

The solution to this equation is similar to the solution for the eqn (8):

$$S(t) = C\exp\left[\kappa\int f(t)dt\right]. \tag{12}$$

Here we have a large variety of models of growth with one of them being the *exponential model* of growth characterised by $f(t) = c$. We might represent $f(t)$ by a polynomial function or by any other function of our choice.

3.3. The logistic model

We might assume that the driving force of growth *decreases with the size* of the growing entity. An example could be the growth of a tree. A tree does not grow indefinitely. It might be growing fast at the beginning but then it reaches a certain average height and does not grow. Growth of an individual person can be also a good example. Initially the growth is fast but eventually a given person reaches a certain height and stops growing. In the simplest case, we might assume that the driving force decreases *linearly* with the size of a growing entity:

$$F(t) = a - bS(t), \tag{13}$$

where a and b are positive constants.

Using the eqn (1) we then have

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = \kappa\left[a - bS(t)\right]. \tag{14}$$

This equation can be expressed as

$$\frac{dS(t)}{\left[A - BS(t)\right]S(t)} = dt\,, \tag{15}$$

where $A \equiv \kappa a$ and $B \equiv \kappa b$.

The left-hand side of this equation can be easily integrated if we split it into a sum of two fractions:

$$\frac{B}{A}\frac{dS(t)}{\left[A - BS(t)\right]} + \frac{1}{A}\frac{dS(t)}{S(t)} = dt\,. \tag{16}$$

From now on, the integration is easy.

Alternatively, we can solve the eqn (15) by using the general integration formula we have derived earlier (Nielsen, 2015):

$$\int \frac{dx}{u \cdot v} = \frac{1}{\Delta}\ln\frac{v}{u}\,, \tag{17}$$

where $u = a + bx$, $v = c + dx$ and $\Delta = ad - bc$.
The solution of the eqn (14) is represented by the sigmoid function:

$$S(t) = \left[\frac{b}{a} + \left(\frac{1}{S_0} - \frac{b}{a}\right)e^{-\kappa a t}\right]^{-1}, \tag{18}$$

where $S_0 = S(t = 0)$.

We can see that

$$S(t \to \infty) \Rightarrow \frac{a}{b} \equiv K\,, \tag{19}$$

where *K* defines the limit of growth.

3.4. The extension of the logistic growth

In the logistic model, the driving force decreases *linearly* to a certain limit *K*. However, we might have many other possibilities. One of them is the modified logistic model introduced by Gilpin & Ayala (1973). Some of the variations to the logistic growth are shown in Figure 1.

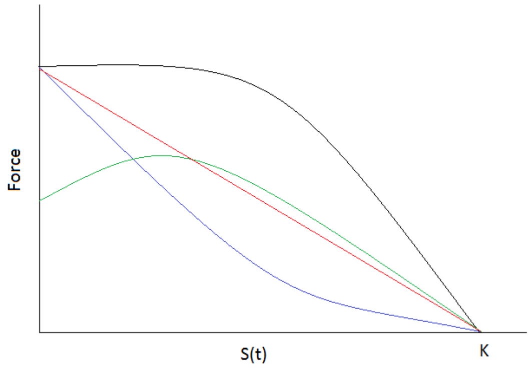

Figure 1. *Examples of the extensions to the conventional logistic growth.*

In Figure 1, the driving force represented by a decreasing straight line to a certain limit *K* represents the well-known, conventional logistic model of growth. However, we might have a force that is initially approximately constant but then is changing gradually to a linearly decreasing. Such a force would describe the initially approximate exponential growth changing seamlessly into a logistic growth, which would be approaching asymptotically the limit *K*.

We could also have a force, which could be initially decreasing rapidly with the size of the growing entity but after a certain time it would start to follow a gently decreasing trajectory. It would be an approximately fast logistic growth changing seamlessly into a slow logistic growth.

Another alternative shown in Figure 1 is a force, which is initially increasing with the size of the growing entity, reaches a certain maximum and then starts to decrease to a certain limit *K*. This type of growth could, for instance, follow an approximately pseudo-hyperbolic trajectory (Nielsen, 2015). However, it would not increase to infinity but it would change seamlessly to an approximately logistic growth, approaching asymptotically the limit *K*.

Possibilities are endless and each of them could be tried to fit data and find their best mathematical representation. However, if we introduce complicated descriptions of the driving force we might have a problem with explaining why we use a complicated description. For instance, if we can see that the growth is indeed initially exponential but then gradually levels off and approaches a certain limit *K*, we could easily describe such a growth mathematically by using a constant driving force changing gradually into the linearly decreasing force shown in Figure 1 but we would still have to explain why the force changed in such a way and why the growth changed from exponential to logistic.

3.5. Further extensions

Even though the general principle in scientific investigations is to use the simplest interpretations, in certain cases it might be necessary to try more complicated solutions and the law of growth offers an easy definition of such more complicated models. The ultimate extension is to assume that the force of growth depends not only on time but also on the size of the growing entity. Such an assumption will probably never be used but it shows that we can have a practically unlimited number of the models of growth.

3.6. The general principle of investigation

Even though the described here general law of growth opens virtually unlimited possibilities for defining and using a wide variety of models of growth, the general principle of scientific investigation is to use the simplest descriptions. In the study of the mechanism of growth the general principle is to use the simplest mechanism of growth as represented by the simplest driving force.

Using complicated mathematical expressions without understanding why we use them and without convincingly justifying their use makes absolutely no sense. Even if complicated expressions lead to a good description of data we have learned nothing about the mechanism of growth unless we can explain why such complicated mathematical descriptions are necessary.

The initial and important step in the study of growth is to identify the *type* of growth. For instance, if we can show that the growth is not exponential but hyperbolic we can then focus our attention on a limited range of forces or maybe even on using just an obvious single force to explain the mechanism of growth. Complicated mathematical descriptions might look impressive, they might create an aura of science, but simple descriptions are always preferable.

4. Summary and conclusion

Using the simplest possible assumption, we have formulated a simple general law of growth. We have shown that this law is analogous to many other simple but useful laws, one of them being the Newton's law of motion. Using a few examples, we have shown how this simple law of growth can be used to define a multiplicity of models of growth, which in turn can be used to study the mechanism of growth. Even though this general law of growth allows for the introduction of a wide variety of models of growth, the general recommendation is to use the simplest descriptions of driving forces to describe and explain the observed phenomena.

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