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## Subsistence-(Threshold) payoff and truncated risk preferences

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**Abstract.** A measure of aversion to a risk akin to the risk premium is the required payoff truncation – a probability level, or a point of the distribution range - of the – null mean - risk distribution that allows an expected utility equal to the deterministic level. For a small risk – a noise of null expected value - added to the argument of an utility function, it is straight-forward to show that – for a risk-averse individual - such subsistence probability equals the conventional risk-premium over the symmetric of the worst possible outcome. However, both measures do not take into account aversion (or proneness) to risk in the utility distribution itself – they apply to expected utility maximizers. Maxmin behaviour and quantile preferences, applicable in the presence of uncertainty (or non-cooperative opponents) rather than risk, can be suggested to circumvent the problem. An alternative theory – constrained expected utility - relies on the use the expected utility over the upper truncated distribution (lower - or doubly truncated - in case of risk-loving behavior) at a given (individual specific) truncation point, or probability level. Then, a conventionally defined risk-premium weighs both the truncation bias and risk dispersion. Such distinction also applies if preference truncation – or rather, "trimming" - is (instead) accompanied by a switch of probability mass to tail "focal" points. Then, if the latter are sufficiently extreme, the effect on attitude towards risk may be reversed relative to standard preference truncation: lower trimming enhancing risk-aversion, upper one reducing it. Applications of truncated principles to mean-variance "utility" preferences – and risk-loving attitudes - were also briefly outlined. Illustrations for normal and uniform risks were often appended.
**Keywords.** Subsistence-payoff; Non-expected utility theories; Constrained expected utility; Truncated preferences towards risk; Maxmin, maxmax; Trimmed preferences towards risk; Focal points; Mean – variance(-utility) preferences; "Trimmed" normal (with tail focal points) distribution; Triangular distribution; Triangular preferences.
**JEL.** D81; C10; C16; C24; D11.

## 1. Introduction

A measure of aversion to a risk akin to the risk premium is the required payoff truncation – a probability level, or a point of the distribution range - of the – null mean - risk distribution that allows an expected utility equal to the deterministic level. For a small risk – a noise of null expected value - added to the argument of an utility function, it is straight-forward to show that – for a risk-averse individual - such subsistence probability equals the risk-premium over the symmetric of the worst possible outcome.

However, both measures do not take into account aversion (or proneness) to risk in the utility distribution itself – they apply to expected utility maximizers. And there is now historical empirical evidence that individuals do not comply with conventional expected utility axioms. Minimax behaviour

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and quantile preferences can be suggested to circumvent the problem: they are used in decision theory under uncertainty – i.e., when the probability distribution is not known – (and in non-cooperative games) rather than risk itself. An alternative theory relies on the use the expected utility over the upper truncated distribution (lower in case of risk-loving behavior) at a given truncation point, or probability level – a concept in some sense compatible with VaR but specified in opposite fashion - of the randomness to represent attitude towards risk. Then, a conventional risk-premium must compensate both the truncation bias and risk dispersion.

The existence of a subsistence level of individual consumption is generally accepted in economic modelling. Likewise, one can adventure that a minimal level of certainty is required for human activity to develop. It can take the form of acceptance of no lotteries but of those truncated from below over the outcome range possibilities; or discarding – disregarding – outcomes outside some range of the distribution. Such constrained expected utility behavior – even if partly consistent with von Neumann-Morgenstern's (VNM) axioms -, as generating corner solutions, can give rise to similar consequences as those of ambiguity-type – prospect or focal point preferences [1] - decision contexts. It also bears realistic features, specially to evaluate firm's behavior under uncertainty: it applies for example to limited liability, a common legal status; insurance schemes often include compensation thresholds suggesting some truncation... On a psychological point of view, it could explain (add to the explanation of... they may just have low probability) why people may find some states surprising even if arising - "lawfully" - from a known risk. It is the purpose of this research to infer some consequences of such optimization behavior.

A different truncation concept involves "trimming" of the original distribution with accumulation of the tail probability masses to individual-specific focal points. Optimization relying on trimming with sufficiently extreme focal points may represent a different deviation of the individual risk-aversion relative to expected utility maximizer than (plainly) truncated preferences: lower (upper) trimming may now induce an increase (reduction) in the risk-premium.

Other decision theories allow for individuals to weigh expected utility along with other risky elements in their objective function; a simple case is the inclusion of the variance of utility itself as argument of the individual's maximand [2]. Constrained mean-variance utility forms were thus also inspected.

Finally, even if directed towards risk, the proposed theories may be generalized to attitudes involving subjective probabilities towards uncertain outcomes [3]. But then the subjective probabilities themselves must be specified. Triangular preferences – see the Appendix –, single-peaked or kinked at a modal point, spanning between two extreme outcomes, would be an alternative to the other special cases inspected.

The exposition is organized as follows: in section 1 advances general notation: risk-premium, and links truncated with "trimmed" moments, in

[1] See Starmer (2000) for a recent survey of non-expected utility theories. Also Rieskamp, Busemeyer and Mellers (2006) for an overview of theories of preferential choice under bounded rationality.

[2] See Martins (2004), for example.

[3] See Kelsey and Quiggin (1992) for a survey.

general and of normal and uniform distributions used in illustrations. Section 2 defines the subsistence payoff (and threshold probability), highlighting connections between it and the standard univariate risk-premium. Section 3 suggests truncated preferences of both risk-averse and risk-loving types, and section 4 further qualifies the risk premium when truncation is accompanied by probability mass accumulation on "focal" tail points. Section 5 considers other non-expected utility extensions. The main conclusions are contained in this introduction.

## 2. Notation

Admit a general (uni-dimensional) function of r attributes, represented by the column vector Z, $\psi(Z)$. The risk premium to a randomness X added to Z is associated to a vector $m_{VNM}$ such that:

$$\psi\,(Z - m_{VNM}) \;=\; E[\psi(Z + X)] \tag{1.1}$$

f(X) denotes the probability density function of X and F(X) the corresponding cumulative distribution function. Let E[X] = $\mu$ = 0, and Cov(X) = V, a symmetric positive semi-definite matrix. Then $m_{VNM}$ stands for a multivariate risk premium defined over the quantities of all the arguments of �(.). It can be specified in the metric of a particular attribute – vector $m_{VNM}$ must embed r-1 linear restrictions for an appropriate definition - if $m_{VNM}$ stands for a vector of zeros except for that attribute.

Considering the Taylor expansion of $\psi(Z - m_{VNM})$ to the first order only:

$$\psi(Z - m_{VNM}) \;\approx\; \psi\,(Z) - \frac{\partial\psi}{\partial Z}\, m_{VNM} \tag{1.2}$$

Replacing in (1.1), we deduce that:

$$\frac{\partial\psi}{\partial Z}\, m_{VNM} \;\approx\; \psi\,[E(Z + X)] - E[\psi(Z + X)] \tag{1.3}$$

$\frac{\partial\psi}{\partial Z}\, m_{VNM}$ – the sum of the elements of vector $m_{VNM}$ weighted by their marginal contribution to the function $\psi$ (.) – is a measure of the difference between the function evaluated at the expected value of the argument and the expected value of the function.

Using Taylor's expansion of E[$\psi$ (Z + X)] to the second order, we infer that

$$\frac{\partial\psi}{\partial Z}\, m_{VNM} \;=\; -\,\frac{1}{2}\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]'\, vec(V) \tag{1.4}$$

where vec(A) creates a column vector stacking the columns of matrix A, $\frac{\partial\psi}{\partial Z}$ and $\frac{\partial^2\psi}{\partial Z\partial Z'}$ the gradient and the Hessian of $\psi(Z)$ respectively. For the common univariate case, $\frac{\partial \psi}{\partial Z}$ m$_{VNM}$ = - $\frac{1}{2}$ $\frac{\partial^2 \psi}{\partial Z^2}$ Var(X), or ψ'(Z) m$_{VNM}$ = - $\frac{1}{2}$ ψ"(Z) $\sigma^2$. We always assume ψ'(Z) > 0.

As is well-known, if e ~ f(e), the truncated distribution at probability levels α (to the left) and β (1 - β at the right) such that $0 \leq \alpha < \beta \leq 1$, exhibits: e | $F^{-1}(\alpha)$ < e < $F^{-1}(\beta)$ ~ $\frac{f(e)}{\beta-\alpha}$ (with cdf $\frac{F(e)-\alpha}{\beta-\alpha}$) over $F^{-1}(\alpha)$ < e < $F^{-1}(\beta)$. ($G^{-1}(.)$ denotes the inverse function of G(.).)

1) Being F(e) a normal cdf with mean $\mu$ and standard deviation $\sigma$ [4]:

$$E[e \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] = \mu + \frac{\phi[\Phi^{-1}(\alpha)] - \phi[\Phi^{-1}(\beta)]}{\beta-\alpha}\sigma \quad (1.5)$$

$$\mathrm{Var}[e \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] = \{1 + \frac{\Phi^{-1}(\alpha)\phi[\Phi^{-1}(\alpha)] - \Phi^{-1}(\beta)\phi[\Phi^{-1}(\beta)]}{\beta-\alpha} - \left[\frac{\phi[\Phi^{-1}(\alpha)] - \phi[\Phi^{-1}(\beta)]}{\beta-\alpha}\right]^2\}\sigma^2 \quad (1.6)$$

and

$$E[e^2 \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] = \{1 + \frac{\Phi^{-1}(\alpha)\phi[\Phi^{-1}(\alpha)] - \Phi^{-1}(\beta)\phi[\Phi^{-1}(\beta)]}{\beta-\alpha}\}\sigma^2 + 2\frac{\phi[\Phi^{-1}(\alpha)] - \phi[\Phi^{-1}(\beta)]}{\beta-\alpha}\sigma\mu + \mu^2 \quad (1.7)$$

$\phi(.)$ and $\Phi(.)$ refer the pdf and cdf of the standard normal respectively. $F^{-1}(\alpha) = \mu + \sigma\,\Phi^{-1}(\alpha)$ and $F^{-1}(\beta) = \mu + \sigma\,\Phi^{-1}(\beta)$.

The normal exhibits, thus, the property that the *bias* in expected value induced by truncation, E[e | $F^{-1}(\alpha)$ < e < $F^{-1}(\beta)$] - E[e], is independent of the mean of the original distribution, $\mu$ (even if not of its variance). Also, the variance of a truncated normal is independent of the mean of the mother pdf.

Also, if $\alpha < \frac{1}{2}$ and $\beta > \frac{1}{2}$, truncation always reduces the variance of a normal pdf – (1.6) is smaller than $\sigma^2$.

2) If e is uniform over the interval (a, b), i.e., e ~ f(e) = $\frac{1}{b-a}$ (F(e) = $\frac{e-a}{b-a}$) over a < e < b, E[e] = $\frac{a+b}{2}$, Var(e) = $\frac{(b-a)^2}{12}$ [5], E[$e^2$] = $\frac{a^2+b^2+ab}{3}$. Then

[4] See, for example, Johnston and Kotz (1970), p. 81.
[5] See, for example, Johnston and Kotz (1970a), p. 59.

the truncated distribution is also uniform, e | $F^{-1}(\alpha) <$ e $< F^{-1}(\beta)$ ~ $\dfrac{f(e)}{\beta-\alpha}$ = $\dfrac{1}{(b-a)(\beta-\alpha)}$ (with cdf $\dfrac{F(e)-\alpha}{\beta-\alpha}$ = $\dfrac{e-[a+(b-a)\alpha]}{(b-a)(\beta-\alpha)}$) over a + (b - a) □ = $F^{-1}(\alpha) <$ e $< F^{-1}(\beta)$ = a + (b - a) □ = b – (b – a) (1 - β) with:

$$E[e \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] = \frac{a+b+(b-a)(\alpha+\beta-1)}{2} \tag{1.8}$$

$$Var[e \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] = \frac{[(b-a)(\beta-\alpha)]^2}{12} = Var(e)\ (\beta-\alpha)^2 \tag{1.9}$$

and

$$E[e^2 \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] = $$
$$= \frac{[a+(b-a)\alpha]^2 + [a+(b-a)\beta]^2 + [a+(b-a)\alpha][a+(b-a)\beta]}{3} \tag{1.10}$$

The triangular density – see definitions and first two moments derived in the Appendix – may also be an alternative.

Truncation – trimming - with focal points can be defined as generating, departing from a mother distribution e ~ f(e), a random variable $e^{(T)}$ | α, $X_L$, β, $X_U$ ~ g[$e^{(T)}$], with the same density as e – g[$e^{(T)}$] = f[$e^{(T)}$] - for $F^{-1}(\alpha) < e^{(T)} < F^{-1}(\beta)$, and concentrating the lower probability mass (of f(e)) before $F^{-1}(\alpha)$, α, on a point $X_L$, the upper (1 - β) after $F^{-1}(\beta)$ on a point $X_U$ – and, of course, $X_L < F^{-1}(\alpha)$ and $X_U > F^{-1}(\beta)$.

It is straightforward to show that the expected value of any function of $e^{(T)}$, h(.):

$$E[h(e^{(T)}) \mid \alpha, X_L, \beta, X_U] = \alpha\, h(X_L) + \int_{F^{-1}(\alpha)}^{F^{-1}(\beta)} h(e)\, f(e)\, de + (1-\beta)\, h(X_U) =$$

$$= \alpha\, h(X_L) + (\beta-\alpha)\, E[h(e) \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] + (1-\beta)\, h(X_U) \tag{1.11}$$

Then, this would apply to E[$e^{(T)}$] itself, and to E[$e^{(T)\,2}$], for example. If f(e) is normal, the replacement of (1.5) and (1.7) in the corresponding form (1.11) would be valid. If uniform, of (1.8) and (1.10).

Also, in general:

$$Var[e^{(T)} \mid \alpha, X_L, \beta, X_U] = E[e^{(T)\,2} \mid \alpha, X_L, \beta, X_U] - E[e^{(T)} \mid \alpha, X_L, \beta, X_U]^2 =$$

$$= \alpha\, X_L^2 + (\beta-\alpha)\, E[e^2 \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] + (1-\beta)\, X_U^2 -$$

$$- \{\alpha\, X_L + (\beta - \alpha)\, E[e \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] + (1 - \beta)\, X_U\}^2 \qquad (1.12)$$

Again, for the normal, the replacement of (1.5) and (1.7) applies. For the uniform, (1.8) and (1.10).

## 3. An alternative definition of risk aversion: Subsistence payoff or probability

Assume that X is uni-dimensional. Likewise, if $\psi(Z + X)$ is concave, we can refer the individual's aversion to risk X (for E[X] = 0, with variance Var(X)) to the probability value $\alpha$ - or, equivalently, the point of the distribution range $F^{-1}(\alpha)$ – at which we have to truncate the distribution f(X) at its left hand-side for him to accept the lottery, i.e., that solves:

$$\psi(Z) = \int_{F^{-1}(\alpha)}^{+\infty} \psi(Z + X)\, \frac{f(X)}{1-\alpha}\, dX = E[\psi(Z + X) \mid X > F^{-1}(\alpha)] \qquad (2.1)$$

An individual may exhibit a larger subsistence payoff (or probability) than another individual for some distribution f(X), but a smaller one for other distribution g(X). An individual would be more risk-averse than another iff his subsistence payoff (or probability) towards any null mean risk is always larger than that of the latter.

Obviously, the three criteria – risk-premium, subsistence payoff and subsistence probability – may generate a different ranking of individuals that face the same risk (then, only subsistence criteria agree with each other), and a different ranking of the risks faced by a given individual. It is therefore important to inspect the properties of the newly advanced measures and, if possible, relate them with the risk premium in at least an approximate manner. We proceed to both throughout this section.

For any truncation level $F^{-1}(\delta)$ of the distribution of X:

$$\frac{dE[\psi(Z+X) \mid X > F^{-1}(\delta)]}{d[F^{-1}(\delta)]} = \frac{f[F^{-1}(\delta)]}{1-\delta} \left\{ - \psi[Z + F^{-1}(\delta)] + \int_{F^{-1}(\delta)}^{+\infty} \psi(Z + X)\, \frac{f(X)}{1-\delta}\, dX \right\} = \frac{f[F^{-1}(\delta)]}{1-\delta} \{E[\psi(Z + X) \mid X > F^{-1}(\delta)] - \psi[Z + F^{-1}(\delta)]\} \qquad (2.2)$$

Obviously, for an increasing function $\psi(.)$, $E[\psi(Z + X) \mid X > F^{-1}(\delta)] > \psi[Z + F^{-1}(\delta)]$ for any truncation level $F^{-1}(\delta)$ and (2.2) is always positive. Given how the truncation level was chosen, around $\delta = \alpha$:

$$\frac{dE[\psi(Z+X) \mid X > F^{-1}(\delta)]}{d[F^{-1}(\delta)]} \Big|_{\delta = \alpha} = \frac{f[F^{-1}(\alpha)]}{1-\alpha} \{\psi(Z) - \psi[Z + F^{-1}(\alpha)]\} \qquad (2.3)$$

We can therefore conclude that for $\alpha$ to exist, $F^{-1}(\alpha) < 0$.

Also, from (2.1):

$$\frac{dE[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta} = \frac{dE[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d[F^{-1}(\delta)]}\ \frac{d[F^{-1}(\delta)]}{d\delta} =$$

$$\frac{dE[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d[F^{-1}(\delta)]}\ \frac{1}{f[F^{-1}(\delta)]} = \frac{1}{1-\delta}\ \{E[\psi(Z+X)\,|\,X>F^{-1}(\delta)] - \psi[Z + F^{-1}(\delta)]\} \tag{2.4}$$

The second derivative is:

$$\frac{d^2E[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta^2} = \frac{1}{1-\delta}\ \{2\ \frac{dE[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta} - \frac{1}{f[F^{-1}(\delta)]}\ \psi'[Z + F^{-1}(\delta)]\} \tag{2.5}$$

Again, evaluated at the susbsistence payoff:

$$\frac{dE[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta}\,|\,\delta=\alpha = \frac{1}{1-\alpha}\ \{\psi(Z) - \psi[Z + F^{-1}(\alpha)]\}$$

$$\frac{d^2E[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta^2}\,|\,\delta=\alpha = \frac{1}{1-\alpha}\ (\ 2\ \frac{1}{1-\alpha}\ \{\psi(Z) - \psi[Z + F^{-1}(\alpha)]\} - \frac{1}{f[F^{-1}(\alpha)]}\ \psi'[Z + F^{-1}(\alpha)]\ )$$

Using Taylor's expansion to the second-order term in ? around 0:

$$E[\psi(Z + X)\,|\,X > F^{-1}(\alpha)] = E[\psi(Z + X)\,|\,X > F^{-1}(0)] + \alpha\ \frac{dE[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta}\,|\,\delta=0 + \frac{1}{2}\ \alpha^2\ \frac{d^2E[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta^2}\,|\,\delta=0 + \ldots = E[\psi(Z + X)] + \alpha\ \frac{dE[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta}\,|\,\delta=0 + \frac{1}{2}\ \alpha^2\ \frac{d^2E[\psi(Z+X)\,|\,X>F^{-1}(\delta)]}{d\delta^2}\,|\,\delta=0 + \ldots =$$

$$= E[\psi(Z + X)] + \alpha\ \{E[\psi(Z + X)] - \psi[Z + F^{-1}(0)]\} + \frac{1}{2}\ \alpha^2\ (2\ \{E[\psi(Z + X)] - \psi[Z + F^{-1}(0)]\} - \frac{1}{f[F^{-1}(0)]}\ \psi'[Z + F^{-1}(0)]) + \ldots \tag{2.6}$$

Then, from the definition of α:

$$\psi(Z) = (1 + \alpha)\, E[\psi(Z + X)] - \alpha\, \psi[Z + F^{-1}(0)] + \frac{1}{2}\, \alpha^2\, (2\, \{E[\psi(Z + X)] - \psi[Z + F^{-1}(0)]\} - \frac{1}{f[F^{-1}(0)]}\, \psi'[Z + F^{-1}(0)]) + \ldots \quad (2.7)$$

With a first-order approximation only [6] – and/or say, ▯$^2$ is relatively negligible:

$$\alpha \approx \frac{\psi(Z) - E[\psi(Z+X)]}{E[\psi(Z+X)] - \psi[Z + F^{-1}(0)]} \quad (2.8)$$

The last expression suggests the importance of the minimum payoff – reminding maximin criteria – as determinant of the magnitude of $\alpha$. It also weighs the expected value of utility and not only the worst possible outcome. Replacing a first-order approximation to the conventional risk premium:

$$\alpha \approx \frac{\psi'(Z)\, m_{VNM}}{\psi(Z) - \psi[Z + F^{-1}(0)] - \psi'(Z)\, m_{VNM}} \quad (2.9)$$

Alternatively, if we expand $E[\psi\,(Z + X)]$ to the second-order in (2.8), we can further infer that:

$$\alpha = -\frac{1}{2}\, \frac{\psi''(Z)\, Var(X)}{\psi(Z) + \frac{1}{2}\psi''(Z)\, Var(X) - \psi[Z + F^{-1}(0)]} \approx -\frac{\psi''(Z)\, \sigma^2}{\psi''(Z)\, \sigma^2 - 2\, \psi'(Z) F^{-1}(0)} \quad (2.10)$$

or, also attending to the risk-premium definition $m_{VNM}$:

$$\frac{\alpha}{1+\alpha} \approx \alpha \approx -\frac{1}{2}\, \frac{\psi''(Z)\, Var(X)}{-\psi'(Z)\, F^{-1}(0)} = \frac{m_{VNM}}{-F^{-1}(0)} \quad (2.11)$$

$$\text{or} \quad m_{VNM} \approx [-F^{-1}(0)]\, \alpha$$

From the first part of (2.10), $\alpha$ is larger the more concave is $\psi(Z)$, and the smaller is the distance between $\psi(Z)$ and the utility of the minimum possible pay-off - the larger the utility derived from the worst outcome. $\alpha$ increases with Var(X), but also with $F^{-1}(0)$ (which is negative).

As $m \approx [-F^{-1}(0)]\, \alpha$, we conclude that *for a given variance Var(X)* distributions with lower $[-F^{-1}(0)]$ – i.e., with a higher minimum of X – will have higher $\alpha$. Therefore, risk-aversion is also influenced by $F^{-1}(0)$ if measured by $\alpha$, while it does not – directly - when assessed by $m_{VNM}$.

[6] These approximations are valid for small risks. For the normal, as $F^{-1}(0) = -$▯, they may not be very useful...

In the reverse angle, $m_{VNM}$ weighs two effects: the threshold probability, α, and (multiplied by) the absolute value of the worst possible outcome, $[- F^{-1}(0)]$.

Using Taylor's expansion to the second-order term - now - in Z we can also approximate:

$$\int_{F^{-1}(\alpha)}^{+\infty} \psi(Z+X)\,\frac{f(X)}{1-\alpha}\,dX = \int_{F^{-1}(\alpha)}^{+\infty} \psi(Z)\,\frac{f(X)}{1-\alpha}\,dX + \int_{F^{-1}(\alpha)}^{+\infty} \psi'(Z)\,X\,\frac{f(X)}{1-\alpha}\,dX + \frac{1}{2}\int_{F^{-1}(\alpha)}^{+\infty} \psi''(Z)\,X^2\,\frac{f(X)}{1-\alpha}\,dX + \ldots = \psi(Z) + \psi'(Z)\int_{F^{-1}(\alpha)}^{+\infty} X\,\frac{f(X)}{1-\alpha}\,dX + \frac{1}{2}\,\psi''(Z)\int_{F^{-1}(\alpha)}^{+\infty} X^2\,\frac{f(X)}{1-\alpha}\,dX + \ldots \quad (2.12)$$

or

$$E[\psi(Z+X) \mid X > F^{-1}(\alpha)] = \psi(Z) + \psi'(Z)\,E[X \mid X > F^{-1}(\alpha)] + \frac{1}{2}\,\psi''(Z)\,E[X^2 \mid X > F^{-1}(\alpha)] + \ldots \quad (2.13)$$

Then, definition (2.1) suggests that ⍰ is the probability at which the ratio between the truncated mean and variance plus squared mean of the exogenous risk X equals $\frac{1}{2}$ of the Arrow (1965)-Pratt (1964)'s measure of absolute risk aversion:

$$\frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]} = \frac{E[X \mid X > F^{-1}(\alpha)]}{Var[X \mid X > F^{-1}(\alpha)] + E[X \mid X > F^{-1}(\alpha)]^2} = \int_{F^{-1}(\alpha)}^{+\infty} X\,\frac{f(X)}{1-\alpha}\,dX \Big/ \left[\int_{F^{-1}(\alpha)}^{+\infty} X^2\,\frac{f(X)}{1-\alpha}\,dX\right] = \int_{F^{-1}(\alpha)}^{+\infty} X\,f(X)\,dX \Big/ \int_{F^{-1}(\alpha)}^{+\infty} X^2\,f(X)\,dX \approx -\frac{1}{2}\,\frac{\psi''(Z)}{\psi'(Z)} \quad (2.14)$$

Then, we can further write that α obeys:

$$\frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]} \approx \frac{m_{VNM}}{Var(X)} \text{ or } m_{VNM} \approx \frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]}\,Var(X) \quad (2.15)$$

For any truncation probability δ,

$$\frac{d\left\{\frac{E[X \mid X > F^{-1}(\delta)]}{E[X^2 \mid X > F^{-1}(\delta)]}\right\}}{d[F^{-1}(\delta)]} = F^{-1}(\delta)\,f[F^{-1}(\delta)]\,\left\{F^{-1}(\delta)\int_{F^{-1}(\delta)}^{+\infty} X\,f(X)\,dX \Big/ \int_{F^{-1}(\delta)}^{+\infty} X^2\,f(X)\,dX - 1\right\} \quad (2.16)$$

At the level δ = α, and because $F^{-1}(\alpha) < 0$:

$$\frac{d\{\frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]}\}}{d[F^{-1}(\alpha)]} = F^{-1}(\alpha)\, f[F^{-1}(\alpha)]\, [- F^{-1}(\alpha)\, \frac{1}{2}\, \frac{\psi''(Z)}{\psi'(Z)} - 1] > 0 \quad (2.17)$$

$\frac{E[X \mid X > F^{-1}(\delta)]}{E[X^2 \mid X > F^{-1}(\delta)]}$ must be increasing with $F^{-1}(\delta)$ and therefore with δ.

*For a particular risk X* – or distribution F(X) -, the higher - $\frac{\psi''(Z)}{\psi'(Z)}$, because α is set such that $\frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]} = - \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)}$, the higher will α be.

Exemplifying with the (null mean) normal distribution – using (1.5) and (1.7) - α (α; β = 1) solves:

$$\frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]} = \frac{\phi[\Phi^{-1}(\alpha)]}{1-\alpha+\Phi^{-1}(\alpha)\phi[\Phi^{-1}(\alpha)]} \frac{1}{\sigma} \approx - \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)} \quad (2.18)$$

With the uniform – using (1.8) and (1.10) and (α; γ = β = 1) - over (a, b) with a = - b:

$$\frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]} = \frac{a+b+(b-a)\alpha}{[a+(b-a)\alpha]^2+b^2+[a+(b-a)\alpha]b} \frac{3}{2} =$$

$$= \frac{2\alpha}{(1-2\alpha)^2 b+2b\alpha} \frac{3}{2} \approx - \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)} \quad (2.19)$$

We can easily verify that for the normal, $\frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]}$ decreases with $\sigma^2$ the parameter of variance for the normal and increases with $F^{-1}(\alpha)$. It decreases with b for the uniform case – and $\frac{E[X \mid X > F^{-1}(\alpha)]}{E[X^2 \mid X > F^{-1}(\alpha)]}$ increases with α. For the equality to be maintained at a given Z, α increases with $\sigma^2$ for the normal and with b for the uniform. Also, α will increase with - $\frac{\psi''(Z)}{\psi'(Z)}$, the Arrow-Pratt measure of risk-aversion.

If ψ(Z + X) is convex, the degree of risk-proneness could be measured by the upper truncation probability mass, β, that would still allow the same utility as complete certainty, i.e., such that:

$$\psi(Z) = \int_{-\infty}^{F^{-1}(1-\beta)} \psi(Z+X) \frac{f(X)}{1-\beta}\, dX = E[\psi(Z+X) \mid X < F^{-1}(1-\beta)] \quad (2.20)$$

The concept can be extended to the multivariate domain measuring the aversion to a risk added to j subject to background noise. The premium in the Arrow-Pratt sense would be the value $n_j$ that solves:

$$E[\psi(Z_1 + X_1, Z_2 + X_2, ..., Z_j - n_j, ..., Z_r + X_r)] = E[\psi(Z + X)] \quad (2.21)$$

Then, provided such $n_j > 0$ and the individual exhibits risk-aversion, an equivalent measure of its degree would be the lower truncation probability $\alpha_j$ – with $F_j(X_j)$ denoting the cumulative marginal distribution of $X_j$ – such that:

$$E[\psi(Z_1 + X_1, Z_2 + X_2, ..., Z_j, ..., Z_r + X_r)] = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \int_{F_j^{-1}(\alpha_j)}^{+\infty} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \psi(Z + X) \frac{f(X)}{1-\alpha_j} dX = E[\psi(Z + X) \mid X_j > F_j^{-1}(\alpha_j)] \quad (2.22)$$

A general multivariate risk-premium can be either formulated as an $\alpha$ such that:

$$\psi(Z) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \int_{F_j^{-1}(\alpha)}^{+\infty} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \psi(Z + X) \frac{f(X)}{1-\alpha} dX = E[\psi(Z + X) \mid X_j > F_j^{-1}(\alpha)] \quad (2.23)$$

One could suggest the minimum (or the maximum…) of the r different $\alpha$'s that could be then calculated – one for each of the r arguments of Z.

Or if all risks have similar dimensions as $\alpha = F(a,a,...,a)$ – or a itself – such that:

$$\psi(Z) = \int_{a}^{+\infty} \int_{a}^{+\infty} ... \int_{a}^{+\infty} \psi(Z + X) \frac{f(X)}{1-F(a,a,...,a)} dX = E\{\psi(Z + X) \mid X > [a\ a\ ...\ a]'\} \quad (2.24)$$

Or considering the marginal distributions, $F_j(X_j)$,

$$\psi(Z) = \int_{F_1^{-1}(\alpha)}^{+\infty} \int_{F_2^{-1}(\alpha)}^{+\infty} ... \int_{F_r^{-1}(\alpha)}^{+\infty} \psi(Z + X) \frac{f(X)}{1-F[F_1^{-1}(\alpha), F_2^{-1}(\alpha), ..., F_r^{-1}(\alpha)]} dX =$$

$$E\{\psi(Z + X) \mid X > [F_1^{-1}(\alpha)\ \ F_2^{-1}(\alpha)\ \ ...\ \ F_r^{-1}(\alpha)]'\} \quad (2.25)$$

Also important is the variance of that conditional variable:

$$Var[\psi(Z + X) \mid X > F^{-1}(\alpha)] = E[\psi(Z + X)^2 \mid X > F^{-1}(\alpha)] - E[\psi(Z + X) \mid X > F^{-1}(\alpha)]^2$$

$$= E[\psi(Z + X)^2 \mid X > F^{-1}(\alpha)] - \psi(Z)^2 \quad (2.26)$$

Suppose a non-expected utility maximizing consumer with deterministic utility function $\psi(Z)$. Suppose $\alpha'$ is his subsistence payoff in face of risk X added to Z; when $\alpha' > \alpha$, the individual is more averse to risk X than the expected utility maximizer. By analogy with (2.1), we can advance that

$$\frac{E[\psi(Z + X) \mid X > F^{-1}(\alpha')]}{E[\psi(Z + X)^2 \mid X > F^{-1}(\alpha')]} \quad (2.27)$$

would contain some measure of aversion to risk in $\psi(.)$. This suggests extensions in line with mean-variance theories, postponed to section 5.

## 3. Truncated Preferences

We can posit that a risk-averse individual maximizes:

$$E[\psi(Z+X) \mid X < F^{-1}(\gamma)] = \int_{-\infty}^{F^{-1}(\gamma)} \psi(Z+X) \frac{f(X)}{\gamma} dX = \psi(Z) + \psi'(Z) E[X \mid X < F^{-1}(\gamma)] + \frac{1}{2} \psi''(Z) E[X^2 \mid X < F^{-1}(\gamma)] + \dots \quad (3.1)$$

He optimizes not expected utility but truncated expected utility. The higher $\gamma$, measuring, along with the shape of his utility function, his optimism - more appropriately, $(1 - \gamma)$ his pessimism -, the more optimistic he is. Obviously, if $F^{-1}(\gamma) = -\infty$ $(\gamma = 0)$, the individual is a Maxmin (utility) character.

A risk-cum pessimism-premium m can then be derived if we consider a noise added to Z such that E[X] = 0, and define m such that:

$$E[\psi(Z+X) \mid X < F^{-1}(\gamma)] = \psi(Z) + \psi'(Z) E[X \mid X < F^{-1}(\gamma)] + \frac{1}{2} \psi''(Z) E[X^2 \mid X < F^{-1}(\gamma)] + \dots = \psi(Z - m) \quad (3.2)$$

Now, we have:

$$m \approx - E[X \mid X < F^{-1}(\gamma)] - \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)} E[X^2 \mid X < F^{-1}(\gamma)] \quad (3.3)$$

The risk-premium now contains two elements: one weighting the change in the expected value of the randomness, the bias, due to truncation, another weighing dispersion; in the latter, variance plus the square bias are included. The decomposition also suggests the independence of the subsistence payoff concept defined in section 2 from the truncation level of preferences… Of course, there will be a subsistence payoff in the spirit of (2.1) equating:

$$\psi(Z) = \int_{F^{-1}(\alpha)}^{F^{-1}[\gamma+\alpha(1-\gamma)]} \psi(Z+X) \frac{f(X)}{(1-\alpha)\gamma} dX = E\{\psi(Z+X) \mid F^{-1}(\alpha) < X < F^{-1}[\gamma + \alpha(1-\gamma)]\} \quad (3.4)$$

The concept becomes more difficult to deal with than the risk-premium.

If X is normal with variance $\sigma^2$ – using (1.5) and (1.7) for $(\nu = \alpha = 0; \gamma = \beta)$:

$$m \approx \frac{\phi[\Phi^{-1}(\gamma)]}{\gamma} \sigma - \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)} \{1 - \frac{\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]}{\gamma}\} \sigma^2 =$$

$$= m_{VNM} + \frac{\phi[\Phi^{-1}(\gamma)]}{\gamma} \sigma + \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)} \frac{\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]}{\gamma} \sigma^2 \quad (3.5)$$

For a VNM entity, $m_{VNM} \approx -\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\sigma^2$. The premium is now increased by the bias, $\frac{\phi[\Phi^{-1}(\gamma)]}{\gamma}\sigma$, but – being ψ(Z) concave and $\gamma > \frac{1}{2}$ so that $\Phi^{-1}(\gamma) > 0$ - attenuated by a change in dispersion effect captured by $\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]}{\gamma}\sigma^2$.

$$m > m_{VNM} \quad \text{iff} \quad -\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\Phi^{-1}(\gamma)\ \sigma < 1 \tag{3.6}$$

If ψ(Z) is concave, the truncated expected value maximizer, when $\gamma > \frac{1}{2}$ and $\Phi^{-1}(\gamma) > 0$, will be more risk-averse than the von Neumann-Morgenstern one when concavity, measured by $-\frac{\psi''(Z)}{\psi'(Z)}$, is relatively mild.

For the interesting special case in which $\gamma = \frac{1}{2}$ and $\Phi^{-1}(\gamma) = 0$, i.e., if individuals maximize E[ψ(Z + X) | X < 0], (3.5) becomes:

$$m \approx 2\ \phi(0)\ \sigma - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\sigma^2 = m_{VNM} + 2\ \phi(0)\ \sigma \tag{3.7}$$

only the bias effect is registered, penalizing (increasing) the risk-premium.

If X is uniform in the interval (a, b) – with a = - b < 0 so that $E[X] = \frac{b+a}{2} = 0$:

$$m \approx -\frac{a+b+(b-a)(\gamma-1)}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{a^2+[a+(b-a)\gamma]^2+a[a+(b-a)\gamma]}{3}$$

$$= \frac{2b(1-\gamma)}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{b^2+[-b+2b\gamma]^2+b[b-2b\gamma]}{3}$$

$$= m_{VNM} - \frac{a+b+(b-a)(\gamma-1)}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\left\{\frac{a^2+[a+(b-a)\gamma]^2+a[a+(b-a)\gamma]}{3} - \frac{(b-a)^2}{12}\right\} =$$

$$= m_{VNM} + \frac{2b(1-\gamma)}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[-b+2b\gamma]^2+b[b-2b\gamma]}{3} =$$

$$= m_{VNM} + \frac{2b(1-\gamma)}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{2b^2(1-2\gamma)(1-\gamma)}{3} = \tag{3.8}$$

Again we devise a bias effect – positive - and a dispersion one – negative for risk-averse individuals if γ is large.

Likewise, a ("truncated") optimist will maximize:

$$E[\psi(Z+X) \mid X > F^{-1}(\nu)] = \psi(Z) + \psi'(Z)\, E[X \mid X > F^{-1}(\nu)] + \frac{1}{2}\, \psi''(Z)\, E[X^2 \mid X > F^{-1}(\nu)] \quad (3.9)$$

with ν being a degree of his optimism. The limiting case $F^{-1}(\nu) = \infty$ ($\nu = 1$) represents a Maxmax utility individual.

Being X normal, the risk-premium becomes ($\nu = \alpha$; $\gamma = \beta = 1$):

$$m \approx - \frac{\phi[\Phi^{-1}(\nu)]}{1-\nu}\, \sigma - \frac{1}{2}\, \frac{\psi''(Z)}{\psi'(Z)}\, \{1 + \frac{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]}{1-\nu}\}\, \sigma^2 =$$

$$= m_{VNM} - \frac{\phi[\Phi^{-1}(\nu)]}{1-\nu}\, \sigma - \frac{1}{2}\, \frac{\psi''(Z)}{\psi'(Z)}\, \frac{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]}{1-\nu}\, \sigma^2 \quad (3.10)$$

$$m < m_{VNM} \text{ iff } - \frac{1}{2}\, \frac{\psi''(Z)}{\psi'(Z)}\, \Phi^{-1}(\nu)\, \sigma < 1 \quad (3.11)$$

If $\psi(Z)$ is concave, the truncated expected value maximizer, when $\nu < \frac{1}{2}$ and $\Phi^{-1}(\nu) < 0$, will always be less risk-averse than the von Neumann-Morgenstern one.

If $\psi(Z)$ is convex, the truncated expected value maximizer will love risk more intensely (m will be more negative) - when $\nu < \frac{1}{2}$ and $\Phi^{-1}(\nu) < 0$ – than the von Neumann-Morgenstern one when convexity, measured by $\frac{\psi''(Z)}{\psi'(Z)}$, is relatively mild.

For the interesting special case in which $\nu = \frac{1}{2}$ and $\Phi^{-1}(\nu) = 0$, i.e., if individuals maximize $E[\psi(Z+X) \mid X > 0]$:

$$m \approx -2\, \phi(0)\, \sigma - \frac{1}{2}\, \frac{\psi''(Z)}{\psi'(Z)}\, \sigma^2 = m_{VNM} - 2\, \phi(0)\, \sigma \quad (3.12)$$

Only the bias further affects the risk-premium, necessarily reducing it relative to the VNM magnitude.

Being X uniform ($\nu = \alpha$; $\gamma = \beta = 1$):

$$m \approx - \frac{a+b+(b-a)\nu}{2} -$$

$$- \frac{1}{2}\, \frac{\psi''(Z)}{\psi'(Z)}$$

$$\frac{[a+(b-a)\nu]^2 + [a+(b-a)]^2 + [a+(b-a)\nu][a+(b-a)]}{3} =$$

$$= -\frac{2b\nu}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[-b+2b\nu]^2+b^2+[-b+2b\nu]b}{3} =$$

$$= m_{VNM} - \frac{2b\nu}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[-b+2b\nu]^2+[-b+2b\nu]b}{3} =$$

$$= m_{VNM} - \frac{2b\nu}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[-b+2b\nu]\,2b\nu}{3} \quad (3.13)$$

A composition of the two effects suggests individuals maximizing $E[\psi(Z + X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)]$. Then:

$$m \approx -\ E[X \mid F^{-1}(\nu) < X < F^{-1}(\gamma)] - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)} E[X^2 \mid F^{-1}(\nu) < X < F^{-1}(\gamma)] \quad (3.14)$$

For the normal ($\nu = \alpha$; $\gamma = \beta$):

$$m \approx -\frac{\phi[\Phi^{-1}(\nu)]-\phi[\Phi^{-1}(\gamma)]}{\gamma-\nu}\sigma -$$

$$- \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\{1+\frac{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]-\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]}{\gamma-\nu}\}\sigma^2 =$$

$$= m_{VNM} - \frac{\phi[\Phi^{-1}(\nu)]-\phi[\Phi^{-1}(\gamma)]}{\gamma-\nu}\sigma - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}$$

$$\frac{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]-\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]}{\gamma-\nu}\sigma^2 \quad (3.15)$$

$m > m_{VNM}$ iff

$$-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]-\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]\}\ \sigma >$$

$$\phi[\Phi^{-1}(\nu)]-\phi[\Phi^{-1}(\gamma)] \quad (3.16)$$

For a symmetric truncation of expected utility, i.e., if $\phi[\Phi^{-1}(\nu)] = \phi[\Phi^{-1}(\gamma)]$, and $\Phi^{-1}(\nu) = -\ \Phi^{-1}(\gamma) < 0$, the bias-correction effect disappears:

$$m = m_{VNM} + \frac{\psi''(Z)}{\psi'(Z)}\frac{\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]}{2\gamma-1}\sigma^2 \quad (3.17)$$

$m < m_{VNM}$ iff $\psi(Z)$ is concave: a symmetric truncation always softens risk-averse behavior. And it also softens risk-loving behavior when $\psi(Z)$ is convex – then, m becomes less negative than $m_{VNM}$.

For the uniform ($\nu = \alpha$; $\gamma = \beta$):

$$m \approx -\frac{a+b+(b-a)(\nu+\gamma-1)}{2} -$$

$$- \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[a+(b-a)\nu]^2+[a+(b-a)\gamma]^2+[a+(b-a)\nu][a+(b-a)\gamma]}{3} =$$

$$= - \frac{2b(\nu+\gamma-1)}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}$$

$$\frac{[-b+2b\nu]^2+[-b+2b\gamma]^2+[-b+2b\nu][-b+2b\gamma]}{3}$$

$$= m_{VNM} - \frac{2b(\nu+\gamma-1)}{2} -$$

$$- \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[-b+2b\nu]^2+[-b+2b\gamma]^2+[-b+2b\nu][-b+2b\gamma]-b^2}{3} \quad (3.18)$$

If we apply symmetric trimming and 1 - γ = ν:

$$m \approx -$$

$$- \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}$$

$$\frac{[a+(b-a)(1-\gamma)]^2+[a+(b-a)\gamma]^2+[a+(b-a)(1-\gamma)][a+(b-a)\gamma]}{3} =$$

$$= - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[-b+2b(1-\gamma)]^2+[-b+2b\gamma]^2+[-b+2b(1-\gamma)][-b+2b\gamma]}{3}$$

$$= m_{VNM} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}$$

$$\frac{[-b+2b(1-\gamma)]^2+[-b+2b\gamma]^2+[-b+2b(1-\gamma)][-b+2b\gamma]-b^2}{3} \quad (3.19)$$

Only the dispersion term remains. For risk averse individuals, the premium may rise with b – but decrease with γ.

Intermediate cases are far from quantile maximization and possess more encompassing value.

## 4. Trimmed Preferences with Tail Focal Points

A similar decision criteria would be provided for the truncation with focal points to which (1.11) applies. Individuals facing randomness X ~ f(X) added to Z with deterministic utility function (Z) would maximize:

$$E[\psi(Z+X^{(T)}) \mid \nu, X_L, \gamma, X_U] =$$

$$= \nu\,\psi(Z+X_L) + \int_{F^{-1}(\nu)}^{F^{-1}(\gamma)} \psi(Z+X)\, f(X)\, dX + (1-\gamma)\,\psi(Z+X_U) =$$

$$= \nu\,\psi(Z+X_L) + (\gamma-\nu)\, E[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)] + (1-\gamma)\,\psi(Z+X_U) \quad (4.1)$$

In general, $X_L$ would be a point in the lower tail, i.e., $X_L < F^{-1}(\nu)$, and $X_U$ in the upper one and $X_U > F^{-1}(\gamma)$.

We could define a pessimist as an individual for whom:

$$X_L < \int_{-\infty}^{F^{-1}(\nu)} X \frac{f(X)}{\nu}\, dX \quad \text{and} \quad X_U < \int_{F^{-1}(\gamma)}^{\infty} X \frac{f(X)}{1-\gamma}\, dX \tag{4.2}$$

This would suggest (even if it would not guarantee) that E[ $X^{(T)}$ | ν, $X_L$, γ, $X_U$] < E[X]. For an optimist, the inequality signs would be reversed. Yet, the (tail) focal points also affect the variance of the distribution.

Alternatively, trimming may just represent "blurred" preferences towards extreme outcomes.

As before, we can infer a risk-premium, call it $m_T$, from:

$$E[\psi(Z + X^{(T)}) \mid \nu, X_L, \gamma, X_U] = \psi(Z - m_T) \tag{4.3}$$

Using (4.1), one immediately concludes that $m_T$ relates to the risk premium of the equivalent truncated preferences m of section 3 as:

$$\psi(Z - m_T) = \nu\, \psi(Z + X_L) + (\gamma - \nu)\, \psi(Z - m) + (1 - \gamma)\, \psi(Z + X_U) \tag{4.4}$$

Then:

$$m_T > m \text{ iff } \psi(Z - m_T) < \psi(Z - m) \text{ or } \psi(Z - m) = E[\psi(Z + X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)]$$
$$> \frac{\nu\, \psi(Z + X_L) + (1-\gamma)\, \psi(Z + X_U)}{\nu + 1 - \gamma} \tag{4.5}$$

We can further expand in (4.4) ψ(Z - m) and ψ(Z - $m_T$) to the first order

$$m_T = (\gamma - \nu)\, m - \frac{\nu\, \psi(Z + X_L) + (1-\gamma)\, \psi(Z + X_U)}{\psi'(Z)} \tag{4.6}$$

Also expanding ψ(Z + $X_L$) and ψ(Z + $X_U$) to the second to conclude that:

$$m_T = (\gamma - \nu)\, m - [\nu X_L + (1 - \gamma) X_U] - \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)} [\nu X_L^2 + (1 - \gamma) X_U^2] =$$
$$= m - [\nu (X_L + m) + (1 - \gamma)(X_U + m)] - \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)} [\nu X_L^2 + (1 - \gamma) X_U^2] \tag{4.7}$$

Assume as before that X is normal. Then m obeys (3.15); replacing it in (4.6):

$$m_T \approx (\gamma - \nu) \left\{ - \frac{\phi[\Phi^{-1}(\nu)] - \phi[\Phi^{-1}(\gamma)]}{\gamma - \nu} \sigma \right. -$$
$$\left. - \frac{1}{2} \frac{\psi''(Z)}{\psi'(Z)} \left\{1 + \frac{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)] - \Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]}{\gamma - \nu}\right\} \sigma^2 \right\} -$$

$$-[\nu X_L+(1-\gamma)X_U]-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}[\nu X_L^2+(1-\gamma)X_U^2] =$$

$$=(\gamma-\nu)\{m_{VNM}-\frac{\phi[\Phi^{-1}(\nu)]-\phi[\Phi^{-1}(\gamma)]}{\gamma-\nu}\sigma-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]-\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]}{\gamma-\nu}\sigma^2\}-$$

$$-[\nu X_L+(1-\gamma)X_U]-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}[\nu X_L^2+(1-\gamma)X_U^2] \tag{4.8}$$

The bias effect is captured by

$$-\{\phi[\Phi^{-1}(\nu)]-\phi[\Phi^{-1}(\gamma)]\}\sigma-[\nu X_L+(1-\gamma)X_U] \tag{4.9}$$

and dispersion by:

$$-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}(\{(\gamma-\nu)+\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]-\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]\}\sigma^2+$$

$$+[\nu X_L^2+(1-\gamma)X_U^2]) =$$

$$=(\gamma-\nu)m_{VNM}-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}(\{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]-\Phi^{-1}(\gamma)\phi[\Phi^{-1}(\gamma)]\}\sigma^2+$$

$$+[\nu X_L^2+(1-\gamma)X_U^2]) \tag{4.10}$$

If X is uniform (b = -a):

$$m_T=(\gamma-\nu)\{-\frac{a+b+(b-a)(\nu+\gamma-1)}{2}-$$

$$-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[a+(b-a)\nu]^2+[a+(b-a)\gamma]^2+[a+(b-a)\nu][a+(b-a)\gamma]}{3}\}$$

$$-\frac{\nu\,\psi(Z+X_L)+(1-\gamma)\,\psi(Z+X_U)}{\psi'(Z)} =$$

$$=(\gamma-\nu)\{-\frac{2b(\nu+\gamma-1)}{2}-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[-b+2b\nu]^2+[-b+2b\gamma]^2+[-b+2b\nu][-b+2b\gamma]}{3}\}$$

$$-\frac{\nu\,\psi(Z+X_L)+(1-\gamma)\,\psi(Z+X_U)}{\psi'(Z)} =$$

$$=(\gamma-\nu)(m_{VNM}-\frac{a+b+(b-a)(\nu+\gamma-1)}{2}-$$

$$-\frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\{\frac{[a+(b-a)\nu]^2+[a+(b-a)\gamma]^2+[a+(b-a)\nu][a+(b-a)\gamma]}{3}-\frac{(b-a)^2}{12}\}) \tag{4.11}$$

$$- \frac{\nu\,\psi(Z+X_L)+(1-\gamma)\,\psi(Z+X_U)}{\psi'(Z)}$$

$$= (\gamma - \nu)\ \{\ m_{VNM}\ -\ \frac{2b(\nu+\gamma-1)}{2}\ -$$

$$-\ \frac{1}{2}\ \frac{\psi''(Z)}{\psi'(Z)}\ \frac{[-b+2b\nu]^2+[-b+2b\gamma]^2+[-b+2b\nu][-b+2b\gamma]-b^2}{3}\}$$

$$- \frac{\nu\,\psi(Z+X_L)+(1-\gamma)\,\psi(Z+X_U)}{\psi'(Z)}$$

We could easily deduct the appropriate expressions for the special cases considered in section 3. They allow us to infer – confirm - the different effect of the single-sided truncation of section 3 from that of the current one: if lower (left hand-side) truncation induced a less risk averse behavior of an individual with a concave utility function, now, truncation with a "focal" lower outcome - $\gamma = 1$ – is likely to enhance – provided $X_L$ is sufficiently negative- his risk-premium. Illustrating with the normal:

$$m_T\ \approx\ -\ \phi[\Phi^{-1}(\nu)]\ \sigma\ -\ \nu\, X_L$$

$$-\ \frac{1}{2}\ \frac{\psi''(Z)}{\psi'(Z)}\ (\{(1 - \nu) + \Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]\}\ \sigma^2\ +\ \nu\, X_L^{\ 2})\ =$$

$$= (1 - \nu)\, m_{VNM} - \phi[\Phi^{-1}(\nu)]\ \sigma\ - \nu\, X_L - \frac{1}{2}\ \frac{\psi''(Z)}{\psi'(Z)}\ \{\Phi^{-1}(\nu)\phi[\Phi^{-1}(\nu)]\ \sigma^2$$

$$+ \nu\, X_L^{\ 2}\} \tag{4.12}$$

For the uniform:

$$m_T\ =\ (1 - \nu)\ \{-\ \frac{a+b+(b-a)\nu}{2}\ -$$

$$-\ \frac{1}{2}\ \frac{\psi''(Z)}{\psi'(Z)}\ \frac{[a+(b-a)\nu]^2+[a+(b-a)]^2+[a+(b-a)\nu][a+(b-a)]}{3}\ \}$$

$$- \frac{\nu\,\psi(Z+X_L)}{\psi'(Z)}\ =$$

$$=\ (1 - \nu)\ \{-\ \frac{2b\nu}{2}\ -\ \frac{1}{2}\ \frac{\psi''(Z)}{\psi'(Z)}\ \frac{[-b+2b\nu]^2+[-b+2b]^2+[-b+2b\nu]b}{3}\}$$

$$- \frac{\nu\,\psi(Z+X_L)}{\psi'(Z)}\ =$$

$$=\ (1 - \nu)\ (\ m_{VNM}\ -\ \frac{a+b+(b-a)\nu}{2}\ -$$

$$-\ \frac{1}{2}\ \frac{\psi''(Z)}{\psi'(Z)}\ \{\ \frac{[a+(b-a)\nu]^2+[a+(b-a)]^2+[a+(b-a)\nu][a+(b-a)]}{3}\ -$$

$$\frac{(b-a)^2}{12}\}\ )$$

$$- \frac{\nu\,\psi(Z+X_L)}{\psi'(Z)}$$

$$= (1 - \nu)\ \{\ m_{VNM} - \frac{2b\nu}{2} - \frac{1}{2}\frac{\psi''(Z)}{\psi'(Z)}\frac{[-b+2b\nu]^2+[-b+2b\nu]b}{3}\} - \frac{\nu\,\psi(Z+X_L)}{\psi'(Z)} \tag{4.13}$$

## 5. Other Non-Expected Utility Extensions

We can depart from individual's behavior relying on the maximization of a mean variance utility function – see Martins (2004): the individual, maximizing in a deterministic context ψ(Z), if faced with a risk X added to Z, maximizes U{E[ψ(Z + X)], Var[ψ(Z + X)]} (instead…).

We can then generalize the criterion to truncated preferences at the two tails. The pertaining risk premium, p, is defined as:

$$U[\psi(Z-p), 0] = U\{E[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)], Var[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)]\} \tag{5.1}$$

By Taylor expansion and if we denote by $U_i(.,.)$ the partial derivative of $U(.,.)$ with respect to the i-th argument and $U_{12}(.,.)$ the cross derivative with respect to the two arguments:

$$U[\psi(Z),0] - U_1[\psi(Z), 0]\ \psi'(Z)\ p = U\{E[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)],0\} + U_2\{E[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)], 0\}\ Var[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)] =$$

$$= U[\psi(Z),0] + U_1[\psi(Z),0]\ \{E[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)] - \psi(Z)\} +$$

$$+ (U_2[\psi(Z),0] + U_{21}[\psi(Z),0]\ \{E[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)] - \psi(Z)\})\ Var[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)]$$ [7]

Noting that $\psi(Z) - E[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)] \approx \psi'(Z)\ m$, where m is the premium of the truncated expected utility maximizer (of section 3):

$$\psi'(Z)\ p = \psi'(Z)\ m + \left(-\frac{U_2[\psi(Z),0]}{U_1[\psi(Z),0]} + \frac{U_{21}[\psi(Z),0]}{U_1[\psi(Z),0]}\ \psi'(Z)\ m\right) Var[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)]$$

Assuming a negligible cross derivative of U(., .):

$$p = m - \frac{1}{\psi'(Z)}\frac{U_2[\psi(Z),0]}{U_1[\psi(Z),0]}\ Var[\psi(Z+X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)] \tag{5.2}$$

[7] Of course, a direct – and more complete - second-order Taylor expansion of the right hand-side would add terms in the square of the variance and in square the of {E[ψ(Z+X)] | ...] - ψ(Z)}. We are assuming that its size is negligible relative to the other terms.

m is the premium of the truncated expected utility maximizer that relates to that of the VNM agent as previously described. Provided $\frac{U_2[\psi(Z),0]}{U_1[\psi(Z),0]} < 0$ – the marginal rate of substitution between the two arguments of U(., .) [8], p > m.

If $\nu < \frac{1}{2}$ and $\gamma > \frac{1}{2}$, the second term, however, may decrease with truncation: that is the expected effect of trimming on the variance of a (for example, for the normal) random variable.

Finally, a further obvious generalization allows for differentiated truncation of the two arguments of the mean-variance utility function, i.e.,

$$U\{E[\psi(Z + X) \mid F^{-1}(\nu) < X < F^{-1}(\gamma)], Var[\psi(Z + X) \mid F^{-1}(\nu') < X < F^{-1}(\gamma')]\} \quad (5.3)$$

Needless to say, trimming with focal points rather than truncation could constrain mean-variance-utility behavior.

[8] It has been identified – see Ormiston and Schlee (2001), Lajeri-Chaherli (2002), Eichner and Wagener (2003) - as the analog to the absolute risk-aversion Arrow-Pratt measure when a function U[E(X), Var(X)] is present.

## Appendix. The Triangular Continuous Distribution

An interesting alternative to the rectangular – uniform – distribution of variable X is the triangular one. It assumes that density, non-null over the (a,b) interval for X, has a peak at c, a < c < b and exhibits a triangular form. It may be important in representing subjective probabilities: when the individual has an idea of what are the worst (a) and best (b) results and the most likelly one (c). Then he assigns linear decay of the density function from the most likely outcome towards those two extreme ones. Graphically:

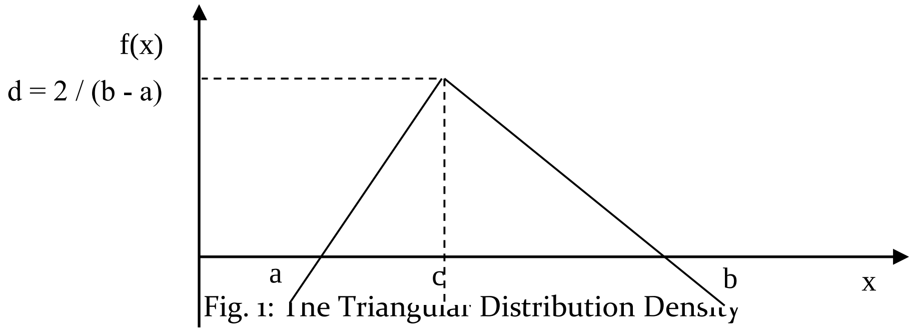


Fig. 1: The Triangular Distribution Density

As the area under (in) the triangle must integrate to 1 = (b – a) f(c) / 2, we conclude that f(c), the ordinate of c, must be f(c) = d = 2 / (b - a). Then, only three parameters – a, b and c -specify the distribution.

For a < x< c, f(x) = y is over the first, increasing line segment that crosses points (a, 0) and (c, 2 / (b – a)); then (c – a) (b – a) / 2 = (x – a) / y or

(A.1) $f(x) = (x - a)\, 2 / [(b - a)(c - a)] \quad a < x < c$

(A.2) $F(x) = \int_a^x f(u)\,du = \int_a^x (u - a)\, 2 / [(b - a)(c - a)]\,du = \{2 / [(b - a)(c - a)]\} \int_a^x (u - a)\,du = \{1 / [(b - a)(c - a)]\} \Big|_a^x u^2 - 2au = \{1 / [(b - a)(c - a)]\}(x^2 - 2ax + a^2) = (x - a)^2 / [(b - a)(c - a)] = F^1(x).$

(A.3) $F^1(c) = (c - a)^2 / [(b - a)(c - a)] = (c - a) / (b - a)$

For c< x < b, f(x) = y is over the decreasing line segment that crosses points (b, 0) and (c, 2 / (b – a)); then (c – b) / [2 / (b – a)] = (x – b) / y or

(A.4) $f(x) = 2(x - b) / [(b - a)(c - b)] = 2(b - x) / [(b - a)(b - c)]\,,\ c < x < b$

(A.5) $F(x) = F^1(c) + \int_c^x f(u)\,du = F^1(c) + \int_c^x (b - u)\{2 / [(b - a) / (b - c)]\}\,du = F^1(c) + \{[2 / [(b - a)(b - c)]\} \int_c^x (b - u)\,du = F^1(c) + \{[1 / [(b - a)(b - c)]\} \Big|_c^x 2bu - u^2 = F^1(c) + \{1 / [(b - a)(b - c)]\}(2bx - x^2 - 2bc + c^2) = (c - a) / (b - a) + \{1 / [(b - a)(b - c)]\}(2bx - x^2 - 2bc + c^2),\quad c < x < b.$

f(x) = 0 otherwise.

(A.6) $E[x] = \int_a^c u\,f(u)\,du + \int_c^b u\,f(u)\,du = \int_a^c (u^2 - au)\, 2 / [(b - a)(c - a)]\,du + \int_c^b 2(bu - u^2) / [(b - a)(b - c)]\,du = \{2 / [(b - a)(c - a)]\} \Big|_a^c u^3 / 3 - au^2 / 2 + \{2 / [(b - a)(b - c)]\} \Big|_c^b bu^2 / 2 - u^3 / 3 =$

$= \{2 / [(b - a)(c - a)]\}(c^3 / 3 - ac^2 / 2 - a^3 / 3 + a^3 / 2) + \{2 / [(b - a)(b - c)]\}(b^3 / 2 - b^3 / 3 - bc^2 / 2 + c^3 / 3)$

(A.7) $E[x^2] = \int_a^c u^2 f(u)\, du + \int_c^b u^2 f(u)\, du = \int_a^c (u^3 - a u^2)\, 2 / [(b-a)(c-a)]\, du + \int_c^b 2 (b u^2 - u^3) / [(b-a)(b-c)]\, du = \{2 / [(b-a)(c-a)]\} \Big|_a^c u^4/4 - a u^3/3 + \{2/[(b-a)(b-c)]\} \Big|_c^b b u^3/3 - u^4/4 =$

$= \{2 / [(b-a)(c-a)]\} (c^4/4 - a c^3/3 - a^4/4 + a^4/3) + \{2/[(b-a)(b-c)]\} (b^4/3 - b^4/4 - b c^3/3 + c^4/4.$

. Symmetry around c, requiring b – c = c – a; replacing then c = (b + a) / 2:

(A.8) $f(x) = 4 (x-a) / (b-a)^2$ , $a < x < (b+a)/2$

(A.9) $f(x) = 4 (b-x) / (b-a)^2$ , $(b+a)/2 < x < b$

(A.10) $E[x] = \int_a^c u f(u)\, du + \int_c^b u f(u)\, du = 4 \int_a^{\frac{a+b}{2}} u (u-a) / (b-a)^2\, du + 4 \int_{\frac{a+b}{2}}^b u (b-u) / (b-a)^2\, du = [4/(b-a)^2] \left( \Big|_a^{\frac{a+b}{2}} u^3/3 - a u^2/2 + \Big|_{\frac{a+b}{2}}^b b u^2/2 - u^3/3 \right) =$

$= [4/(b-a)^2] ([(a+b)/2]^3/3 - a [(a+b)/2]^2/2 - a^3/3 + a^3/2) + (b^3/2 - b^3/3 - b [(a+b)/2]^2/2 + [(a+b)/2]^3/3)$

(A.11) $E[x^2] = 4 \int_a^{\frac{a+b}{2}} u^2 f(u)\, du + 4 \int_{\frac{a+b}{2}}^b u^2 f(u)\, du = 4 \int_a^{\frac{a+b}{2}} u^2 (u-a) / (b-a)^2\, du + 4 \int_{\frac{a+b}{2}}^b u^2 (b-u) / (b-a)^2\, du = [4/(b-a)^2] \left( \Big|_a^{\frac{a+b}{2}} u^4/4 - a u^3/3 + \Big|_{\frac{a+b}{2}}^b b u^3/3 - u^4/4 \right) =$

$= [4/(b-a)^2] ([(a+b)/2]^4/4 - a [(a+b)/2]^3/3 - a^3/3 + a^4/4 + (b^4/3 - b^4/4 - b [(a+b)/2]^3/3 + [(a+b)/2]^4/4)$

. Finally, with symmetry around 0, a = - b and:

(A.12) $f(x) = (x+b)/b^2$ , $-b < x < 0$

(A.13) $f(x) = (b-x)/b^2$ , $0 < x < b$

(A.14) $F(x) = (x^2/2 + b x)/b^2 - (b^2/2 - b^2)/b^2 = (x^2/2 + b x)/b^2 + 1/2$, $-b < x < 0$

(A.15) $F(x) = 1/2 + (b x - x^2/2)/b^2$ , $0 < x < b$

(A.16) $E[x] = 0$

(A.17) $E[x^2] = Var(X) = \int_{-b}^0 u^2 f(u)\, du + \int_0^b u^2 f(u)\, du = \int_{-b}^0 u^2 (u+b)/b^2\, du + \int_0^b u^2 (b-u)/b^2\, du = (1/b^2) \left( \Big|_{-b}^0 u^4/4 + b u^3/3 + \Big|_0^b b u^3/3 - u^4/4 \right) =$

$= (1/b^2)(-b^4/4 + b^4/3 + b^4/3 - b^4/4) = b^2/6$

The truncated version of the previous special case will exhibit ($\alpha < \frac{1}{2}$ and $\beta > \frac{1}{2}$)

(A.18) $f(x) = (x + b) / [b^2 (\beta - \alpha)]$, $F^{-1}(\alpha) < x < 0$

(A.19) $f(x) = (b - x) / [b^2 (\beta - \alpha)]$, $0 < e < F^{-1}(\beta)$ ]

(A.20) $F(x) = \{[(x^2 / 2 + b x) / b^2 + 1 / 2 - \alpha\} / (\beta - \alpha)$, $F^{-1}(\alpha) < x < 0$

(A.21) $F(x) = \{1 / 2 + [(b x - x^2 / 2) / b^2] - \alpha\} / (\beta - \alpha)$, $0 < x < F^{-1}(\beta)$

$F^{-1}(\alpha)$ (for $\alpha < 1 / 2$) is the x that solves $[(x^2 / 2 + b x) / b^2 + 1 / 2 = \alpha$; $F^{-1}(\beta)$ (for $\beta > 1 / 2$) is the x that solves $1 / 2 + [(b x - x^2 / 2) / b^2] = \beta$.

(A.22) $$F^{-1}(\alpha) = b\,(-1 \pm \sqrt{1-2\alpha}) = b\,(-1 + \sqrt{1-2\alpha}) > -b$$

(A.23) $$F^{-1}(\beta) = b\,(1 \pm \sqrt{2-2\beta}) = b\,(1 - \sqrt{2-2\beta}) < b$$

To simplify the expressions, we do not replace the solution in later expressions.

(A.24) $$E[x \mid F^{-1}(\alpha) < x < F^{-1}(\beta)] = \int_{F^{-1}(\alpha)}^{0} u\, f(u) / (\beta - \alpha)\, du + \int_{0}^{F^{-1}(\beta)} u\, f(u) / (\beta - \alpha)\, du$$

$$= \int_{F^{-1}(\alpha)}^{0} u\,(u + b) / [b^2 (\beta - \alpha)]\, du + \int_{0}^{F^{-1}(\beta)} u\,(b - u) / [b^2 (\beta - \alpha)]\, du = \{1 / [b^2 (\beta - \alpha)]\} \left( \Big|_{F^{-1}(\alpha)}^{0} u^3 / 3 + b\, u^2 / 2 + \Big|_{0}^{F^{-1}(\beta)} b\, u^2 / 2 - u^3 / 3 \right) =$$

$$= \{1 / [b^2 (\beta - \alpha)]\} \left(-F^{-1}(\alpha)^3 / 3 - b\, F^{-1}(\alpha)^2 / 2 + b\, F^{-1}(\beta)^2 / 2 - F^{-1}(\beta)^3 / 3\right)$$

(A.25) $$E[x^2 \mid F^{-1}(\alpha) < x < F^{-1}(\beta)] = \int_{F^{-1}(\alpha)}^{0} u^2 f(u) / (\beta - \alpha)\, du + \int_{0}^{F^{-1}(\beta)} u^2 f(u) / (\beta - \alpha)\, du = \{1 / [b^2 (\beta - \alpha)\} \int_{F^{-1}(\alpha)}^{0} u^2 (u + b)\, du + \{1 / [b^2 (\beta - \alpha)\} \int_{0}^{F^{-1}(\beta)} u^2 (b - u)\, du = \{1 / [b^2 (\beta - \alpha)\} \left( \Big|_{F^{-1}(\alpha)}^{0} u^4 / 4 + b\, u^3 / 3 + \Big|_{0}^{F^{-1}(\beta)} b\, u^3 / 3 - u^4 / 4 \right) =$$

$$= \{1 / [b^2 (\beta - \alpha)]\} \left[-F^{-1}(\alpha)^4 / 4 - b\, F^{-1}(\alpha)^3 / 3 + b\, F^{-1}(\beta)^3 / 3 - F^{-1}(\beta)^4 / 4\right]$$

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