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Multivariate risk exposure: Risk-premium, optimal decisions and mean-variance implications

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Abstract. This research develops and expands the concept of risk-premium to a multivariate environment, providing an operational framework for the analysis of mean-variance optimizers' attitudes towards exogenous uncertainty. Firstly, it digresses over possible approximations to the risk premium. Secondly, importance and properties of the variance of the objective function are highlighted. Thirdly, impact of uncertainty on the objective function and on control variables of mean-variance agents is confronted with that of expected function optimizer's. The analysis is also applied to ex-post flexible or adjustable environments with respect to the decision variables. Production theory examples are briefly sketched.Innovation in tools include matrix algebra results and representation of higher than second moments – with reference to the multinormal as a special case -, and implicit rules of first-order condition point-wise optimization of functions of expected value and of variance of other functions.

Keywords. Multivariate uncertainty: Multivariate risks; Risk-premium and risk-aversion; Background noise; Firm's valuation; Mean-variance; Commitment under uncertainty; The value of information/flexibility; Uncertainty and the firm; Matrix algebra; Matrix vectorization and differentiation; Kronecker product; Multivariate normal distribution.

JEL. D80; D21; L14; L15; G11; G12; C60; C61; C69; C10.

1. Introduction

Multivariate analysis in the theory of uncertainty is highly technical and often redounds in unintuitive outcomes. It is the purpose of this research to contribute to the understanding of its mechanics and frameworking.

Even in the univariate domain, where the role of concavity of the objective function is graphically understood, the quantitative measurement of the response to uncertainty only becomes perceptible through the mathematical development of the properties of the risk-premium – of how much of a given asset or income is the individual willing to forego to avoid the randomness. The risk-premium provides a measure of the impact of uncertainty on the expected value of a given function in the metric of one of its arguments. Through its inspection, the Arrow (1965) and Pratt (1964)'s absolute (and relative) measure of risk-aversion measure emerge as conditioning the magnitude of passive impact on expected utility, Kimball's (1990) prudence of the effect of risk on control/decision variable of an optimizing agent, Gollier & Pratt's (1996) temperance and Martins' (2004) providence assessing background uncertainty.

On the other hand, von Neumann-Morgenstern agents – expected function maximizers –are not the only prototypes simulating individual's behavior in the presence of uncertainty dealt with in the economics literature. Non-

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expected utility theories count recent applications in resolving empirical paradoxes (Starmer, 2000). In the finance area, the mean-variance approach – see Tobin (1958) and Markowitz (1959) (Allais, 1979 as cited in Starmer, 2000[1])-, that encompasses the eclectic treatment as a special case, is probably the most well-known, with relevance in asset-pricing formation research (Sharpe, 1694; Linther, 1965; Black, 1972) among others. Applications in production theory have also followed (Karni & Schmeidler, 1991). Its contrast with expected utility preferences has been the subject of recent studies in risk and insurance theory – Ormiston & Schlee (2001), Lajeri-Chaherli (2002) and Eichner & Wagener (2003).

Naturally, an inquiry into the properties and adequate definition of a risk-premium under the assumption would stand as useful, and its multivariate generalization as fundamental - and became the main goal of this article. Historically, it continues the sequel of Duncan's (1977), Karni's (1979), Kihlstrom, Romer & Williams' (1981) (also Keeney, 1973) and others' work, searching for an appropriate multivariate risk representation – for von Neumann-Morgenstern agents.

Under multiple variable interaction, matrix representation, with more compact outcomes than the underlying summations, products and others, becomes useful. Yet, notation and properties of its algebra do not seem to have had a consistent use in mathematical applications. A first task was to develop theorems applicable to the analysis, mostly on matrix differentiation rules involving vectorization and Kronecker products – honouring Dhrymes' (1978) matrix calculus legacy. Among others, a tractable Taylor's expansion form – invariably essential in risk theory approximations - was derived; and third and fourth moment matrix representations for the multivariate normal.

An application of the principles yielded the representation of the expected value but also of the variance of a function of uncertain multiple, possibly correlated arguments. The development of the latter is important for the understanding of the impact of exogenous variability on the behavior of a mean-variance entity. Importance of higher-order derivatives and moments of the exogenous randomness(es) distribution becomes visible – without reliance on higher than second-order expansions, subject explored for the bivariate case in Martins (2004), for example.

Features of optimal decisions become more complex under uncertain environments. The subject has been studied in microeconomic consumption and production theory; general conclusions can only be derived with a multivariate representation which we were set to inspect. We staged two scenarios – constant controls decided before the realization of the random event; and ex-post decision-making. If ex-ante commitment implies control variable stability – with optimal decisions completely sterilizing indirect effects of uncertainty on the objective function -, ex-post flexibility offers the potential to use the control variables in order to reduce the actual (total) "direct" maximand's fluctuations.

Ex-post flexibility in the control variables would get the expected-value mazimizer back to the exogenous uncertainty background, now referred to a deterministic optimal – optimized – indirect problem. If the randomness(es) is (are) added to the decision variables, it turns the expected-function

[1] -proposed a model in which individuals' preferences "may also depend on the second moment of utility, that is, the variance of utility about the mean". One can say that some of the former theories propose preferences over the mean and variance of a certain random variable.

mazimizer into a deterministic optimizer on the expected value – it allows for the neutralization of the effect of any risk. That may not be the case for a mean-variance agent. Moreover, in some contexts, even if no other defence is available, point-wise pure discarding of utility may be a meaningful option and, if capable of being sufficiently variance diminishing, have a place in optimal planning of the latter.

The exposition is organized as follows: in section 1, we advance general notation and develop expected value and variance equivalences. Section 2, digresses over operational definitions of the risk-premium in the multivariate case. Section 3 explores the properties of optimal controls under uncertain backgrounds. Section 4 generates analogous conclusions for ex-post adjustable decision contexts. Section 5 advances general statements on the implications of combining the several backgrounds. Some applications to production theory are noted in section 6. The exposition ends with some concluding remarks. (Theorems of matrix algebra are compiled in Appendix 1, Taylor's expansion in vector form advanced in Appendix 2, multivariate normal moment matrices developed in Appendix 3.)

2. Notation: Multivariate Risk Exposure and Moments of Multi-Argument Function

Admit a general (uni-dimensional) function of r attributes, represented by the column vector Z, $\psi(Z)$. We adopt Dhrymes (1978) conventions with respect to matrix operations – they are stated in Appendix 1.

Consider a column vector X of dimension r. Using Taylor's expansion – see Duncan (1977) -, $\psi(Z + X)$ can be approximated by:

$$\psi(Z+X) = \psi(Z) + \frac{\partial\psi}{\partial Z} X + \frac{1}{2!} X' \frac{\partial^2\psi}{\partial Z \partial Z'} X + \ldots \quad (1)$$

$\frac{\partial\psi}{\partial Z}$ is the row-vector with r elements containing the first derivatives of $\psi(Z)$ with respect to each of the r Zi's – it is the gradient of $\psi(Z)$. denotes the (symmetric) Hessian matrix of $\psi(Z)$, the matrix of second derivatives.

Let X denote an r-dimensional multivariate random variable, of mean E[X] = μ and variance-covariance (symmetric) matrix Cov(X) = E[(X - μ) (X - μ)'] = E[X X'] - $\mu\mu'$ = V; μ_i denotes the element of the i-th row of μ; μ_{ij}, the element in the i-th row and j-th column of V. dμ denotes the column vector of differentials of the several μ_i's. Also dvec(V) is a rrx1 column vector containing the rxr differentials of the variances and covariances of X; of course, when assessing effects of out of the diagonal terms of V, one has to add two of dvec(V)'s factoring elements.

It is easily established that, provided the elements of X are small:

Proposition 1: $E[\psi(Z+X)] \approx \psi(Z) + \frac{\partial\psi}{\partial Z}\mu + \frac{1}{2}\left[vec\left(\frac{\partial^2\psi}{\partial Z \partial Z'}\right)\right]' vec(V + \mu\mu') =$

$$= \psi(Z) + \frac{\partial\psi}{\partial Z}\mu + \frac{1}{2} tr[\frac{\partial^2\psi}{\partial Z \partial Z'}(V + \mu\mu')]$$

Proof: Denote $\frac{\partial \psi}{\partial Z}$ by G (a row-vector with r elements) and $\frac{\partial^2 \psi}{\partial Z \partial Z'}$ by H (a symmetric square matrix of order r). Taking the expectation of (1.1), only the last term, involving E[X' $\frac{\partial^2 \psi}{\partial Z \partial Z'}$ X] = E[X' H X] would not be obvious. X' H X is a scalar, hence equal to its trace. E[X' H X] = E[tr(X' H X)]; as tr(A B) = tr(B A) as long as operations are conformable, E[tr(X' H X)] = E[tr(H X X')] = tr(H E[X X']) = tr[H (V + μ μ')]. Using Proposition A.3 of Appendix 1, and noting that H symmetric:

$$E[X' H X] = vec(H)' vec(V + \mu \mu') = vec(V + \mu \mu')' vec(H) \quad (2)$$

We can deduce that:

$$\frac{\partial E[\psi(Z+X)]}{\partial vec(V)} \approx \frac{1}{2}\left[vec\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)\right]' \quad (3)$$

Notice that the effect of an exogenous change in the level of the deterministic arguments Z on expected utility is given by (using Proposition A.5 in the Appendix 1):

$$\frac{\partial E[\psi(Z+X)]}{\partial Z} = \frac{\partial \psi}{\partial Z} + \mu' \left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right) + \frac{1}{2}\frac{\partial tr\left[\frac{\partial^2 \psi}{\partial Z \partial Z'}(V+\mu\mu')\right]}{\partial Z} =$$

$$= \frac{\partial \psi}{\partial Z} + \mu' \left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right) + \frac{1}{2} vec(V+\mu\mu')' \frac{\partial\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)}{\partial Z} \quad (4)$$

However, with the same order approximation we only capture the first two terms when assessing a change in μ:

$$\frac{\partial E[\psi(Z+X)]}{\partial \mu} = \frac{\partial \psi}{\partial Z} + \mu' \left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right) \quad (5)$$

Proof: Using the rule of the derivative of the trace of the product rule of Proposition A.8 in Appendix 1, letting A = $\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)$, X = μ', B = 1, α = μ, we recover that $\frac{\partial tr\left[\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)\mu\mu'\right]}{\partial \mu} = 2\mu' \left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)$.

Third derivatives condition (1.4); second ones (1.5). Yet, the two effects should coincide under infinite (full) approximations.

Also of interest would be the variance of the function. Ignoring higher than second-order terms, taking the covariance of the right hand-side of (1.1), we conclude - using the fact that if a is a constant and x and y random variables, Var(x + y + a) = Var(x) + Var(y) + 2 Cov(x, y):

$$\mathrm{Var}[\psi(Z+X)] \approx \mathrm{Var}(G\,X) + \frac{1}{4}\,\mathrm{Var}(X'\,H\,X) + \mathrm{Cov}(G\,X,\; X'\;H\,X)$$

One can show that:

Proposition 2: $\mathrm{Var}[\psi(Z+X)] \approx \frac{\partial\psi}{\partial Z}\, V\, \frac{\partial\psi}{\partial Z'} +$

$$+ \frac{1}{4}\left(\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]' E[(XX')\otimes(XX')]\; vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right) - \{[\mathrm{vec}(V + \mu\mu')]'\; vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\}^2\right) +$$

$$+ \frac{\partial\psi}{\partial Z}\, E[X'\otimes(XX')]\; vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right) - \frac{\partial\psi}{\partial Z}\,\mu\,[\mathrm{vec}(V+\mu\mu')]'\, vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right) =$$

$$= \mathrm{Var}[\psi(Z+X)] \approx \left[vec\left(\frac{\partial\psi}{\partial Z'}\frac{\partial\psi}{\partial Z}\right)\right]' \mathrm{vec}(V) +$$

$$+ \frac{1}{4}\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]' E[(XX')\otimes(XX')] - \mathrm{vec}(V + \mu\mu')[\mathrm{vec}(V + \mu\mu')]'\}\; vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right) +$$

$$+ \frac{\partial\psi}{\partial Z}\;\{E[X'\otimes(XX')] - \mu\,[\mathrm{vec}(V+\mu\mu')]'\}\;\; vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)$$

Proof: The first term has a trivial correspondence: Var(G X) = G Var(X) G' if G is deterministic. We can use the fact that tr(A B) = tr(B A) and Proposition A.3 in Appendix 1 to develop the same first term in the second correspondence.

The second term can be developed in the following way:

Var(X' H X) = E[X' H X X' H X] - E[X' H X]2. Using (1.2), we can recognize the squared term. E[X' H X X' H X] = E[tr(X' H X X' H X)] = E[tr(X X' H X X' H)]. Using the trace of the product rule of Proposition A.4 of Appendix 1, we can derive that E[tr(X X' H X X' H)] = E{vec(H)' [(XX')⊗(XX')] vec(H)} = vec(H)' E[(XX')⊗(XX')] vec(H).

As for the third term, Cov(G X, X' H X) = E{(G X – G μ)(X' H X - E[X' H X])} = E{G X (X' H X - E[X' H X])} - G μ E(X' H X - E[X' H X])} = G E[X X' H X] – G μ E[X' H X])}. E[X' H X] is given on (1.2). E[G X X' H X] = E[tr(G X X' H X)] =

E[tr(X X' H X G)]; applying again the trace of the product rule, E[tr(X X' H X G)] = E{vec(H)' [(XX')⊗X] vec(G)} = E{vec(G')' [X'⊗(XX')] vec(H)}. Also, as G is a row vector, vec(G') = vec(G) = G':

$$E[G X X' H X] = vec(H)' E[(XX')\otimes X] G' = G E[X'\otimes(XX')] vec(H) \quad (6)$$

However - see Proposition E.4 of Appendix 2:

$$E\{(X - \mu)'\otimes[(XX' - E(XX')]\} \neq E[X'\otimes(XX')] - \mu [vec(V + \mu\mu')]'$$

Third centered moments are related to the asymmetry or skewness in the distribution of X – so, also that matrix, but in a more distant correspondence. It is easily shown that for a null expected value multivariate normal - symmetric around zero - that E[X'⊗(XX')] = 0.
Also - see Proposition E.7 of Appendix 2:

$$E\{[XX'-E(XX')]\otimes[(XX'-E(XX')]\} \neq E[(XX')\otimes(XX')] - vec(V+\mu\mu')[vec(V+\mu\mu')]'$$

One can now deduce, using Propositions A.1, A.5 and A.7 of Appendix 1 that:

$$\frac{\partial Var[\psi(Z+X)]}{\partial vec(V)} \approx \left[vec\left(\frac{\partial\psi}{\partial Z'}\frac{\partial\psi}{\partial Z}\right)\right]' +$$

$$+ \frac{1}{4} vec[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]']' \{\frac{\partial E[(XX')\otimes(XX')]}{\partial vec(V)} -$$

$$- [vec(V + \mu\mu') \otimes I_{rr}] - [I_{rr} \otimes vec(V + \mu\mu')]\} +$$

$$+ vec[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\frac{\partial\psi}{\partial Z}]' \{\frac{\partial E[X\otimes(XX')]}{\partial vec(V)} - (\mu \otimes I_{rr})\} \quad (7)$$

For the zero mean multivariate normal – see Proposition G.4 in Appendix 3 - the last term disappears and we are left with:

$$\frac{\partial Var[\psi(Z+X)]}{\partial vec(V)} \approx \left[vec\left(\frac{\partial\psi}{\partial Z'}\frac{\partial\psi}{\partial Z}\right)\right]' +$$

$$+ \frac{1}{4} vec[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]']' \{\frac{\partial E[(XX')\otimes(XX')]}{\partial vec(V)} -$$

$$- [vec(V) \otimes I_{rr}] - [I_{rr} \otimes vec(V)]\} \quad (8)$$

$\frac{\partial E[(XX')\otimes(XX')]}{\partial vec(V)}$ can be computed from Proposition G.5 in Appendix 3.
Sensitivity to Z implies the development of higher order differentiation (using Proposition A.6 of Appendix 1):

$$\frac{\partial Var[\psi(Z+X)]}{\partial Z} = 2\,\frac{\partial\psi}{\partial Z}\,V\,\frac{\partial^2\psi}{\partial Z\partial Z'} +$$

$$+\ \frac{\partial\psi}{\partial Z}\ \{E[X'\otimes(XX')] - \mu\,[vec(V+\mu\mu')]'\}\ \frac{\partial\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]}{\partial Z} +$$

$$+\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]'\ \{E[X'\otimes(XX')] - \mu\,[vec(V+\mu\mu')]'\}'\ \frac{\partial^2\psi}{\partial Z\partial Z'} +$$

$$+\ \frac{1}{2}\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]'\ \{E[(XX')\otimes(XX')] - vec(V+\mu\mu')[vec(V+\mu\mu')]'\}\ \frac{\partial\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]}{\partial Z} \qquad (9)$$

2. Multivariate Risk-Premium

2.1. von-Neumann-Morgenstern Multivariate Risk-Premium: Definitions

Consider Proposition 1. Admit that $\psi(Z)$ is positively related to any of its arguments. It easily follows that we can define the column vector m such that:

$$\psi(Z - m) = E[\psi(Z+X)] \qquad (2.1)$$

Let $E[X] = 0$. Then m stands for a multivariate risk premium defined over the quantities of all the arguments of $\psi(.)$. Considering the Taylor expansion of $\psi(Z - m)$ to the first order only:

$$\psi(Z - m) \approx \psi(Z) - \frac{\partial\psi}{\partial Z}\,m \qquad (2.2)$$

Replacing in (2.1), we deduce that:

$$\frac{\partial\psi}{\partial Z}\,m \approx \psi[E(Z+X)] - E[\psi(Z+X)] \qquad (2.3)$$

$\frac{\partial\psi}{\partial Z}\,m$ – the sum of the elements of vector m weighted by their marginal contribution to the function $\psi(.)$ – is a measure of the difference between the function evaluated at the expected value of the argument and the expected value of the function.

Replacing Proposition 1, we infer that

$$\frac{\partial \psi}{\partial Z}\ \mathrm{m} = -\frac{1}{2}\left[vec\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)\right]' \mathrm{vec(V)} \qquad (2.4)$$

As it stands, several m's are compatible with the equation. According to the settings, we can re-define m in one of the arguments of Z – say, a risk-less asset -, i.e., let m = [0 0 ... m_i 0 ... 0]'. Then:

Proposition 3: The premium to general multivariate risks
1. can be defined in the metric of a particular asset as:

$$m_i = -\frac{1}{2}\left(\frac{\partial \psi}{\partial Z_i}\right)^{-1}\left[vec\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)\right]' \mathrm{vec(V)} \qquad (2.5)$$

2. reacts to variances and covariances according to:

$$\frac{\partial m_i}{\partial \sigma_{jk}} = -\frac{\dfrac{\partial^2 \psi}{\partial Z_j \partial Z_k}}{\dfrac{\partial \psi}{\partial Z_i}} \ \text{if } j \neq k; \qquad \frac{\partial m_i}{\partial \sigma_{jj}} = -\frac{1}{2}\frac{\dfrac{\partial^2 \psi}{\partial Z_j^{\,2}}}{\dfrac{\partial \psi}{\partial Z_i}} \qquad (2.6)$$

We recognize in (2.6) the roles of the Arrow-Pratt measure of absolute risk aversion – "absolute concavity" - of ψ(Z), $-\dfrac{\dfrac{\partial^2 \psi}{\partial Z_j^{\,2}}}{\dfrac{\partial \psi}{\partial Z_i}}$, and of $-\dfrac{\dfrac{\partial^2 \psi}{\partial Z_j \partial Z_k}}{\dfrac{\partial \psi}{\partial Z_i}}$ - measuring "absolute substitutability" between Z_j and Z_k in function ψ(Z), given that a high (positive) $\dfrac{\partial^2 \psi}{\partial Z_j \partial Z_k}$ suggests complementarity between the two arguments, inspected by Duncan (1977), Karni (1979) and Martins (2004) -, determining the impact of the effect of changes in the second moments of the distribution of X on the size of the risk-premium.

Alternatively, we could re-define the risk premium as the scalar v such that m = v [1 1 ... 1]' = v L, where L denotes the column vector [1 1 ... 1]' – it implies a decrease v in the certain consumption of all goods simultaneously that would leave the consumer indifferent to the actual randomness he faces.

$$\mathrm{v} = -\frac{1}{2}\left(\frac{\partial \psi}{\partial Z} L\right)^{-1}\left[vec\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)\right]' \mathrm{vec(V)} \qquad (2.7)$$

Then:

$$\frac{\partial v}{\partial \sigma_{jk}} = - \frac{\frac{\partial^2 \psi}{\partial Z_j \partial Z_k}}{\frac{\partial \psi}{\partial Z} L} \quad \text{if } j \neq k; \quad \frac{\partial v}{\partial \sigma_{jj}} = - \frac{1}{2} \frac{\frac{\partial^2 \psi}{\partial {Z_j}^2}}{\frac{\partial \psi}{\partial Z} L} \tag{2.8}$$

An alternative view of risk aversion can be inferred if, following the decomposition of Proposition 1, if we look at the trade-off between elements of and elements of V that sustain a given – fixed – expected utility level. Considering (1.3) and (1.5), we can write:

$$0 = \frac{\partial \psi}{\partial Z} \ d\mu + \mu' \left(\frac{\partial^2 \psi}{\partial Z \partial Z'} \right) d\mu + \frac{1}{2} \left[vec\left(\frac{\partial^2 \psi}{\partial Z \partial Z'} \right) \right]' \text{dvec(V)}$$

that is:

$$[\frac{\partial \psi}{\partial Z} + \mu' \left(\frac{\partial^2 \psi}{\partial Z \partial Z'} \right)] \, d\mu = - \frac{1}{2} \left[vec\left(\frac{\partial^2 \psi}{\partial Z \partial Z'} \right) \right]' \text{dvec(V)} \tag{2.9}$$

With a second-order approximation, if we only consider the effect of the change in one μ_i - say, the/a risk-less asset -, it will depend on the means of the other X's. It is immediate to conclude that:

Proposition 4: The sensitivity of an agent towards uncertainty can be ascertained by the trade-off measuring how much he must be given in expected value of a given commodity to accept an increase in the moments of the random variables distribution,

1. defined as:

$$d\mu_i = - \frac{1}{2} \left(\frac{\partial \psi}{\partial Z_i} + \mu' \frac{\partial \psi}{\partial Z' \partial Z_i} \right)^{-1} \left[vec\left(\frac{\partial^2 \psi}{\partial Z \partial Z'} \right) \right]' \text{dvec(V)} \tag{2.10}$$

2. reacting to particular moments according to:

$$\frac{\partial \mu_i}{\partial \sigma_{jk}} = - \frac{\frac{\partial^2 \psi}{\partial Z_j \partial Z_k}}{\frac{\partial \psi}{\partial Z_i} + \mu' \frac{\partial^2 \psi}{\partial Z' \partial Z_i}} \quad \text{if } j \neq k; \quad \frac{\partial \mu_i}{\partial \sigma_{jj}} = - \frac{1}{2} \frac{\frac{\partial^2 \psi}{\partial {Z_j}^2}}{\frac{\partial \psi}{\partial Z_i} + \mu' \frac{\partial^2 \psi}{\partial Z' \partial Z_i}} \tag{2.11}$$

The denominator of (2.11) appears more complex than in (2.6), but the role of the numerator remains unaltered. Moreover, if we evaluate the trade-off around $\mu = 0$, the two expressions coincide.

Of course, more complex approximations – using expansion to higher order as in Appendix 2 – would generate more refined definitions. Then attention should be given to third and fourth moments, as performed for the bivariate case in Martins (2004), for example. Then, the equivalence of the two definitions evaluated at $\mu = 0$ may not hold.

A final contrast with the premium to a risk j when subject to background noise can be made. Using only Taylor's expansion, such premium to a risk, say, X_j added to Z_j, denoted by n_j, would be such that:

$$E[\psi(Z_1 + X_1, Z_2 + X_2, \ldots, Z_j - n_j, \ldots, Z_r + X_r)] = E[\psi(Z + X)] \quad (2.12)$$

Denote by Z-j the (r-1)x1 vector containing all other elements of Z except Z_j; V_j the (r-1)x1 vector containing the j-th column of V to the exception of line j, i.e., of $\sigma_{jj} - V_j' = [\ \sigma_{1j}\ \sigma_{2j}\ \ldots\ \sigma_{j-1,j}\ \sigma_{j+1,j}\ \ldots\ \sigma_{rj}\ ]'$; and V-j the covariance matrix of X-j, the vector containing all the elements of X but Xj. By analogy with (2.3), we infer now that:

$$\frac{\partial\psi}{\partial Z_j}\, n_j \approx E[\psi(Z_{-j} + X_{-j}, E(Z_j + X_j)] - E[\psi(Z + X)] \quad (2.13)$$

$\frac{\partial\psi}{\partial Z_j} n_j$, the partial premium nj weighted by its marginal contribution to ψ (.), measures the difference between the expected value of the function over the r-1 arguments evaluated at the expected value of $Z_j + X_j$ and the (general) expected value of the function.

Expanding and decomposing both sides of (2.12) - allowing matrix partition for the right hand-side -, as the terms $\frac{1}{2}\left[vec\left(\frac{\partial^2\psi}{\partial Z_{-j}\partial Z_{-j}'}\right)\right]' vec(V_{-j})$ cut, we would arrive at:

$$n_j = -\frac{1}{2}\left(\frac{\partial\psi}{\partial Z_j}\right)^{-1}\left(\frac{\partial^2\psi}{\partial Z_j^2}\sigma_{jj} + 2\frac{\partial^2\psi}{\partial Z_j \partial Z_{-j}} V_j\right) \quad (2.14)$$

Relying on Taylor's expansion to a second-order approximation only, due to its polynomial properties, nj responds only to the r σ_{jk}'s, k=1,2,...,r, but in the same fashion as the global multivariate premium defined in the metric of Z_j, m_j, would[2], i.e.:

$$\frac{\partial n_j}{\partial\sigma_{jk}} = \frac{\partial m_j}{\partial\sigma_{jk}} \text{ for } k = 1, 2, \ldots, r \quad ; \quad \text{but } \frac{\partial n_j}{\partial\sigma_{lk}} = 0 \text{ for any } l, k \neq j \quad (2.15)$$

We would have that:

$$m_j = \left(\frac{\partial\psi}{\partial Z_j}\right)^{-1}\left\{\sum_{i=1}^{r}\frac{\partial\psi}{\partial Z_i} n_i + \frac{1}{2}\left\{vec\left[\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)_d\right]\right\}' vec(V_d)\right\} \quad (2.16)$$

[2] That may not be hold if we use higher-order Taylor's expansion approximations and (or) higher than second-order moment matrices (moments) of the distribution of X depend on (the elements of) V. This would be the case for a multivariate normal, for example – see Martins (2004).

where V_d and $\left(\frac{\partial^2\psi}{\partial Z \partial Z'}\right)_d$ stand for V and $\frac{\partial^2\psi}{\partial Z \partial Z'}$ respectively with the diagonal elements replaced by 0´s.

The expression suggests that the maximizer will more likely insure the whole joint risks rather than one at a time – he is more negatively affected by the whole, in terms of expected value, than by the sum of the partial risks (he is made better-off by discarding the whole risks simultaneously rather than each of them unilaterally) and $m_j \frac{\partial\psi}{\partial Z_j} > \sum_{i=1}^{r} \frac{\partial\psi}{\partial Z_i} n_i$ - for:

- positively correlated risks around arguments that are complements, i.e., for which $\frac{\partial^2\psi}{\partial Z_k \partial Z_l} > 0$.

- negatively correlated risks around arguments that are substitutes, i.e., for which $\frac{\partial^2\psi}{\partial Z_k \partial Z_l} < 0$.

Identical conclusions would be driven from setting in (2.9) all elements of $d\mu$ but $d\mu_j$, and in dvec(V) all but those elements in dVj to 0 – and evaluating the expression at $\mu = 0$.

In this research, we concentrate on the role of a global risk-premium.

2.2. Mean-Variance Compatible Risk-Premium

Under mean-variance approaches, agents respond to the expected value of a function but also to its variance. Potentially, they maximize, say, $U\{E[\psi(Z+X)], Var[\psi(Z+X)]\}$. We will denote the first partial derivative of U(.,.) with respect to the first argument by U1(.,.), to the second by U2(.,.) and the second partial derivatives in accordance.

Consider a standard consumer and let Z be univariate, representing income, with X having null mean. A von Neumann-Morgenstern expected utility function expanded to the second order would imply:

$$E[\psi(Z+X)] = \psi(Z) + \frac{1}{2}\frac{\partial^2\psi}{\partial Z^2} Var(X) \qquad (2.17)$$

If the consumer maximizes expected utility, he cares about $E[Z+X] = Z$ – positively, provided $\frac{\partial\psi}{\partial Z} > 0$ -, and about the $Var(Z+X) = Var(X)$. If he is risk-averse, $\frac{\partial^2\psi}{\partial Z^2} < 0$ and he obviously reacts negatively to the latter. Hence, a truly mean-variance behavior of a von Neumann Morgenstern individual towards (Z + X) is suggested by the right hand-side of (2.17).

One can say that mean-variance approaches generalize the reasoning made towards (Z + X) to the function $\psi(Z+X)$ itself, and (but) frees any connection between the impact of the mean and of the variance[3]: admit optimization is oriented by a function of U{E[ψ (Z + X)], Var[ψ(Z + X)]}, potentially embedding more or less risk aversion than just E[ψ (Z + X)] accommodates – or than an hypothetical representation E{G[ψ (Z + X)]}, with G(.) being a particular function [4], would (which would still be a von Neumann-Morgenstern case). It is useful for production theory where a profit function ψ(Z + X) of several arguments and measured in money metrics is empirically meaningful, but utility derived from the several consumers/investors is not. A "direct" risk-premium g, would obey:

$$U\{E[\psi(Z+X)] - g, 0\} = U\{E[\psi(Z+X)], Var[\psi(Z+X)]\} \qquad (2.18)$$

Expanding the left hand-side in the first argument around E[ψ(Z + X)] to the first-order:

$$U\{E[\psi(Z+X)] - g, 0\} = U\{E[\psi(Z+X)], 0\} - U_1\{E[\psi(Z+X)], 0\}\, g$$

In line with (2.3) and (2.13) we could write:

$$U_1\{E[\psi(Z+X)], 0\}\, g = U\{E[\psi(Z+X)], 0\} - U\{E[\psi(Z+X)], Var[\psi(Z+X)]\} \qquad (2.19)$$

g when weighted by the marginal utility with respect to the first argument affers the difference betweem the utility function evaluated at zero variance and at its actual value.

Expanding also the right hand-side of (2.18) in the second argument around 0, admitting Var[ψ(Z + X)] to be small, we derive:

$$g \approx - \frac{U_2\{E[\psi(Z+X)], 0\}}{U_1\{E[\psi(Z+X)], 0\}} Var[\psi(Z+X)]$$

$$\text{or: } g \approx - \frac{U_2\{E[\psi(Z+X)], 0\} Var[\psi(Z+X)] + \frac{1}{2} U_{22}\{E[\psi(Z+X)], 0\}\{Var[\psi(Z+X)]\}^2}{U_1\{E[\psi(Z+X)], 0\}} \qquad (2.20)$$

Interestingly, if we only take first-order approximations, g is dependent of E[ψ(Z + X)], and, at a given value of it, proportional to Var[ψ(Z + X)]. If ultimately, the ramdomness X is determining the variance of ψ(Z + X),

[3] Under (2.17), if $\frac{\partial \psi}{\partial Z}$ is the impact of an unitary increase of the mean, of the variance must be $\frac{1}{2}\frac{\partial^2 \psi}{\partial Z^2}$.

[4] Even if this presided to Tobin (1958)'s derivation - relying on a probability distribution dependent on the mean and the variance of the argument of the function the expected value of which was maximized.

provided $\frac{U_2\{E[\psi(Z+X)],0\}}{U_1\{E[\psi(Z+X)],0\}}$ is invariant to $E[\psi(Z+X)]$, the determinants of $Var[\psi(Z+X)]$ condition the risk-premium in a similar pattern.

The expression also suggests why $-\frac{U_2\{E[\psi(Z+X)],Var[\psi(Z+X)]\}}{U_1\{E[\psi(Z+X)],Var[\psi(Z+X)]\}}$, (minus) the marginal rate of substitution between the second and first arguments of U(.,.) has been identified – see Ormiston & Schlee (2001), Lajeri-Chaherli (2002), Eichner & Wagener (2003)[5] - as the analog to the absolute risk-aversion Arrow-Pratt measure[6]. Under the current scenario, such definition becomes insufficient:

Even if $\frac{U_2\{E[\psi(Z+X)],0\}}{U_1\{E[\psi(Z+X)],0\}}$ was constant, g cannot be assumed proportional to the risk-premium of a von Neumann-Morgenstern agent that reacts to higher order moments – say, uses Taylor expansion to the 4-th order -, once the functional relations would be much changed. That is, g should compare with

$$\psi(Z) - E[\psi(Z+X)] \approx \frac{\partial\psi}{\partial Z}\, m \approx -\frac{1}{2}\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]' vec(V)$$

where m denotes the (a) EU agent premium vector of (2.4). Admitting only a first order importance - and independence of $\frac{U_2\{E[\psi(Z+X)],0\}}{U_1\{E[\psi(Z+X)],0\}}$ from $E[\psi(Z+X)]$ -, changes in V affect the mean-variance utility at the rate of the square of first derivatives of $\psi(Z+X)$ – as we can infer from (1.7) and (1.8) -, whereas for the von Neumann-Morgenstern agent, the first effects are weighted by second derivatives of $\psi(Z+X)$.

To compare both risk-premia, redefine it in the new utility function in the metric of Z as the rx1 vector p:

$$U[\psi(Z-p), 0] = U\{E[\psi(Z+X)], Var[\psi(Z+X)]\} \quad (2.21)$$

The current definition would also incorporate the fact that a null variance of $E[\psi(Z+X)]$ – present in the left hand-side - may only be achieved through a constant X = 0. Developing the left hand-side to the first order we conclude:

$$U_1\{\psi[E(Z+X)], 0\}\, \frac{\partial\psi}{\partial Z}\, p = U\{\psi[E(Z+X)], 0\} - U\{E[\psi(Z+X)], Var[\psi(Z+X)]\}$$

[5] Only Lajeri-Chaherli constitutes the variance as second argument of the mean-variance utility function, the other authors using the standard-deviation instead. This latter approach becomes more tractable when analyzing preferences over portfolio composites, once some form of invariance to proportional changes is directly preserved with it. For our purposes, the former is more convenient.

[6] In fact, for a "direct mean-variance" – with the variance by second argument - over a univariate random variable, its size would be comparable – see (2.11) - with that of $\frac{1}{2}$ the Arrow-Pratt absolute measure of the alternative univariate utility function of the classical expected utility maximizer

(2.22)

Developing also the righ hand-side to the first order:

$$U[\psi(Z),0] - U_1[\psi(Z), 0] \frac{\partial \psi}{\partial Z} p = U\{E[\psi(Z+X)],0\} + U_2\{E[\psi(Z+X)], 0\} Var[\psi(Z+X)] =$$

$$= U[\psi(Z),0] + U_1[\psi(Z),0] \{E[\psi(Z+X)] - \psi(Z)\} +$$

$$+ (U_2[\psi(Z),0] + U_{21}[\psi(Z),0] \{E[\psi(Z+X)] - \psi(Z)\}) Var[\psi(Z+X)]^7$$

Noting that $\psi(Z) - E[\psi(Z+X)] \approx \frac{\partial \psi}{\partial Z}$ m:

$$\frac{\partial \psi}{\partial Z} p = \frac{\partial \psi}{\partial Z} m + (-\frac{U_2[\psi(Z),0]}{U_1[\psi(Z),0]} + \frac{U_{21}[\psi(Z),0]}{U_1[\psi(Z),0]} \frac{\partial \psi}{\partial Z} m) Var[\psi(Z+X)] \quad (2.23)$$

Beyond the risk aversion embedded in the concavity of $\psi(Z)$, there will be now the “direct” effect captured in the second argument of the MV utility function U(.,.). Then, considering a particular asset to define the premium, and p = [0 0 ... pi 0 ... 0]’, we conclude:

Proposition 5: 1. The risk-premium of a “mean-variance” agent will relate to a von Neumann-Morgenstern’s according to:

$$p_i = m_i + \{-\left(\frac{\partial \psi}{\partial Z_i}\right)^{-1} \frac{U_2[\psi(Z),0]}{U_1[\psi(Z),0]} + \frac{U_{21}[\psi(Z),0]}{U_1[\psi(Z),0]} m_i\} Var[\psi(Z+X)] \quad (2.24)$$

2. The trade-off with expected value of a relative commodity could be expressed as $d\eta i$, relating to that of the expected function maximizer, $d\mu i$, as:

$$d\eta_i = d\mu_i + \left(\frac{\partial \psi}{\partial Z_i} + \mu' \frac{\partial \psi}{\partial Z' \partial Z_i}\right)^{-1} \{- \frac{U_2[\psi(Z),0]}{U_1[\psi(Z),0]} + \frac{U_{21}[\psi(Z),0]}{U_1[\psi(Z),0]} \frac{\partial \psi}{\partial Z_i} d\mu_i\} dVar[\psi(Z+X)] \quad (2.25)$$

There will be an added term to compensate relative to the von Neumann-Morgenstern entity. Being U21[ψ(Z),0] negligible, such term is positive provided $\frac{U_2[\psi(Z),0]}{U_1[\psi(Z),0]} < 0$, and at given Z or for a constant, influenced in an approximately proportional fashion by Var[ψ(Z+X)].

Of course, Var[ψ(Z+X)] depends also on the moments of the distribution of X, including second moments as noted in Proposition 2. Ultimately, risk-aversion is dictated by how the elements of V influence pi after such correspondence – and that of m_i through (2.5) - is internalized (replaced in

[7] Of course, a direct – and more complete - second-order Taylor expansion of the right hand-side would add terms in the square of the variance and in square the of {E[ψ(Z+X)] - ψ(Z)}. We are assuming that its size is negligible relative to the other terms.

(2.24)), as well as any impact of V in higher (third or fourth) moments of the particular distribution of X.

Some clarifying words about the mean-value formulation above – and that will be studied in this research – should be added:

Firstly, we remind (and caution) the reader that the utility function U{E[ψ(Z + X)], Var[ψ(Z + X)]} is a mean-variance utility function towards ψ(Z + X). By comparing it with E[ψ(Z + X)], we are in fact contrasting the corresponding agent with a risk-neutral von Neumann- Morgenstern entity towards that same argument – or, in general, of form E[ν(Z + X)].

Secondly, hypothetically, a generalized multivariate "mean-variance" unit could be forwarded as a maximizer of U[E(Z + X), Cov(Z + X)] = U(Z + μ, V), where we conform with previous notation – E[X] = μ, Cov(X) = V. Inspection of its properties will be pursued elsewhere.

Finally, and as a theoretical contribution to the modeling of individual behaviour towards risk – multivariate or not -, one studies the formulation U{E[ψ(Z + X)], Var[ψ(Z + X)]}, a multivariate "mean-variance utility" utility function, an alternative to the standard expected utility - E[ψ(Z + X)] – maximizer, being ψ(Z) the equivalent function maximized in the absence of uncertainty. Such behavioral hypothesis was used before in economic modelling – the use of higher moments of utility was previously proposed by Allais (1979) and Hagen (1979), cited in Starmer (2000): in this research, some of its consequences are inspected.

3. Optimal Decisions under Uncertain Background

3.1. The Multivariate Conditions under Ex-Ante Commitment

Under certain contexts, the vector Z may be controllable. An expected value maximizing entity will choose Z such that (1.4) is set to zero (we admit E[X] = 0)[8] :

$$\frac{\partial E[\psi(Z+X)]}{\partial Z} = \frac{\partial \psi}{\partial Z} + \frac{1}{2}\frac{\partial tr\left(\frac{\partial^2\psi}{\partial Z\partial Z'}V\right)}{\partial Z} = 0$$

$$\text{or}\quad \frac{\partial \psi}{\partial Z} + \frac{1}{2}\text{vec(V)'}\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z} = 0\,;\ \frac{\partial \psi}{\partial Z'} + \frac{1}{2}\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z'}\text{vec(V)} = 0 \quad (3.1)$$

The mean-variance agent chooses Z such that:

$$U_1\{E[\psi(Z+X)], Var[\psi(Z+X)]\}\frac{\partial E[\psi(Z+X)]}{\partial Z} + $$

$$+\ U_2\{E[\psi(Z+X)], Var[\psi(Z+X)]\}\frac{\partial Var[\psi(Z+X)]}{\partial Z} = 0 \qquad (3.2)$$

[8] We might have as well considered a departure from the expansion of the functions in the vector $\frac{\partial \psi(Z+X)}{\partial Z}$ around Z, take its expected value and perform E[$\frac{\partial \psi(Z+X)}{\partial Z}$] = 0, deriving conclusions henceforth. It appeared as a less tractable format.

The marginal rate of substitution between the two arguments of U(.,.) is equated to the symmetric of the ratio of the elements of $\frac{\partial E[\psi(Z+X)]}{\partial Z}$ by the analogous ones of . That is, the Z's are leveled in such a way that for any i:

$$\frac{\frac{\partial E[\psi(Z+X)]}{\partial Z_i}}{\frac{\partial Var[\psi(Z+X)]}{\partial Z_i}} = -\frac{U_2\{E[\psi(Z+X)],Var[\psi(Z+X)]\}}{U_1\{E[\psi(Z+X)],Var[\psi(Z+X)]\}} \qquad (3.3)$$

Admit that Z is univariate. If U2 < 0, as long as $\frac{\partial Var[\psi(Z+X)]}{\partial Z} > 0$, U(., .) is already decreasing with the argument, Z, at the point chosen by the expected function maximizer – at $\frac{\partial E[\psi(Z+X)]}{\partial Z} = 0$, (3.2) is negative. Then, the mean variance agent chooses a smaller Z.

Proposition 6: The "mean-variance" agent (with U2 < 0) is expected to choose:

1. Lower levels of the deterministic controls, Z, if (for which) $\frac{\partial Var[\psi(Z+X)]}{\partial Z} > 0$
2. Higher levels of the deterministic controls, Z, if (for which) $\frac{\partial Var[\psi(Z+X)]}{\partial Z} < 0$

than the von Neumann-Morgenstern one.

From the decomposition (1.9) and for the univariate case, if the effect of the first term in the right hand-side of (1.9) dominates, we conclude for the second case provided that $\frac{\partial \psi}{\partial Z} > 0$ and $\frac{\partial^2 \psi}{\partial Z \partial Z'} < 0$ – i.e., $\frac{\partial Var[\psi(Z+X)]}{\partial Z} < 0$.

Take a univariate distribution. If increases with an exogenous parameter α, the optimal Z of an expected utility maximizer (second-order conditions ensure a negative second derivative of E[ψ(Z+X)] with respect to Z) increases with α.

For example, consider a change in the covariance matrix elements. The change in the optimal decisions will conform with (3.1) and obey:

$$\{2\frac{\partial^2 \psi}{\partial Z \partial Z'} + [vec(V)' \otimes I_r] \frac{\partial \left[\frac{\partial\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)}{\partial Z'}\right]}{\partial Z}\}\, dZ + \frac{\partial\left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)}{\partial Z'}\, dvec(V) = 0$$

or $$dZ = -\left\{2\frac{\partial^2\psi}{\partial Z\partial Z'} + [vec(V)' \otimes I_r]\frac{\partial\left[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z'}\right]}{\partial Z}\right\}^{-1}\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z'}\, dvec(V) \qquad (3.4)$$

The sign effect of the change in a single element of vec(V), $d\sigma_{ij}$ or $d\sigma_{jj}$, on Z, is given by (using Proposition A.5 of Appendix 1):

$$dZ = -2\left\{2\frac{\partial^2\psi}{\partial Z\partial Z'} + [vec(V)' \otimes I_r]\frac{\partial\left[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z'}\right]}{\partial Z}\right\}^{-1}\frac{\partial\left(\frac{\partial^2\psi}{\partial Z_i\partial Z_j}\right)}{\partial Z'}\, d\sigma_{ij} \quad \text{if } i \neq j;$$

$$dZ = -\left\{2\frac{\partial^2\psi}{\partial Z\partial Z'} + [vec(V)' \otimes I_r]\frac{\partial\left[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z'}\right]}{\partial Z}\right\}^{-1}\frac{\partial\left(\frac{\partial^2\psi}{\partial Z_{jj}^2}\right)}{\partial Z'}\, d\sigma_{jj} \qquad (3.5)$$

That is, the effect on the optimal factor k, dZ_k, is determined by the elements of the column vector $\frac{\partial\left(\frac{\partial^2\psi}{\partial Z_i\partial Z_j}\right)}{\partial Z'}$ or $\frac{\partial\left(\frac{\partial^2\psi}{\partial Z_{jj}^2}\right)}{\partial Z'}$, weighted by the elements of the k-th row of $A = \left\{2 + [vec(V)' \otimes I_r]\frac{\partial\left[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z'}\right]}{\partial Z}\right\}^{-1}$:

$$dZ_k = -2\sum_{l=1}^{r} a_{kl}\frac{\partial^3\psi}{\partial Z_i\partial Z_j\partial Z_l}\, d\sigma_{ij} \quad \text{if } i \neq j\text{ ; } \quad dZ_k = -\sum_{l=1}^{r} a_{kl}\frac{\partial^3\psi}{\partial Z_{jj}^2\partial Z_l}\, d\sigma_{jj} \qquad (3.6)$$

This is consistent with Kimball (1990) assessment of the importance of the measure of absolute prudence, weighting third-order derivatives and conditioning the impact of uncertainty on the control variables themselves. Notice also that A (or its inverse) must be negative-definite for (3.1) to guarantee a maximum.

For the mean variance entity, a more complicated requirement is imposed. If $U2 < 0$, if $\frac{\partial E[\psi(Z+X)]}{\partial Z}$ increases (decreases) with α and $\frac{\partial Var[\psi(Z+X)]}{\partial Z}$ decreases (increases) with α, Z will likely increase (decrease) with α - provided the effects weighted by the second derivatives of U are small). If $\frac{\partial E[\psi(Z+X)]}{\partial Z}$ and $\frac{\partial Var[\psi(Z+X)]}{\partial Z}$ react in the same way to α, the sign effect may be positive or negative, depending on the size of U1 and U2 that weight each of the two cross derivatives (and of second derivatives).

Due to the requirement $\frac{\partial E[\psi(Z+X)]}{\partial Z}$ = 0, the indirect impact of uncertainty, i.e., of vec(V) on the maximal expected utility becomes zero and the total effect simple to derive – it coincides with (1.3), measured at the optimal controls: in any of the two cases:

Proposition 7: The effect of uncertainty on the maximand of an entity with (ex-ante) control over exogenous variables is:

1. indistinguishable from that of an exogenous effect of a change in the distribution of X on the relevant maximand.

2. assessable in a symmetric way by the numerator of the conventional risk-premium definition, by the premium itself if a particular metric is called for its evaluation

3.2. Mean-Variance Opportunity Frontier

A meaningful intermediate decision problem of the mean-variance agent would determine vector Z that minimizes Var[ψ(Z + X)] subject to a certain E[ψ(Z + X)] is achieved. Or vice-versa. That is, solve:

$$\underset{Z}{Min}\ \ Var[\psi(Z+X)]$$

$$\text{s.t.:}\qquad E[\psi(Z+X)] \geq \pi \tag{3.7}$$

or equivalently in lagrangean form

$$\underset{Z,\lambda}{Min}\ \ L(Z,\lambda) = Var[\psi(Z+X)] + \lambda\,\{\pi - E[\psi(Z+X)]\} \tag{3.8}$$

where λ denotes the multiplier. F.O.C. imply:

$$\frac{\partial L}{\partial Z} = \frac{\partial Var[\psi(Z+X)]}{\partial Z} - \lambda\,\frac{\partial E[\psi(Z+X)]}{\partial Z} = 0 \text{ (a (1 x n) vector)} \tag{3.9}$$

$$\frac{\partial L}{\partial \lambda} = \pi - E[\psi(Z+X)] = 0 \text{ (a scalar)} \tag{3.10}$$

Admit the approximation $\frac{\partial E[\psi(Z+X)]}{\partial Z}$ Z ≈ u E[ψ(Z + X)], where u denotes a constant (for linear functions ψ(Z), it is 1; for concave functions, it may be represented by a value smaller than 1) – a measure of the elasticity of the expected value with respect to the control variables (if all Z_i's increase by x%, E[ψ(Z + X)] would rise – proportionately - u x%) – or the returns to scale of E[ψ(Z + X)] with respect to Z. Then, in the optimal solution:

$$\lambda^* = \frac{1}{u\pi} \frac{\partial Var[\psi(Z+X)]}{\partial Z} Z$$

Replacing in (3.9),

$$\frac{\partial Var[\psi(Z+X)]}{\partial Z} = \frac{1}{u\pi} \frac{\partial Var[\psi(Z+X)]}{\partial Z} Z \frac{\partial E[\psi(Z+X)]}{\partial Z}. \text{ Then:} \quad (3.11)$$

$$\frac{\partial Var[\psi(Z+X)]}{\partial Z} \{ Z \frac{\partial E[\psi(Z+X)]}{\partial Z} - u\,\pi\, I_r \} = 0 \quad (3.12)$$

Z is set in such a way that (u π) is an eigenvalue of the left hand-side matrix; as the latter, being the product of a vector by its transpose, has rank 1, Z will be such that (u π) will be the unique non-zero eigenvalue of Z $\frac{\partial E[\psi(Z+X)]}{\partial Z}$ and $\frac{\partial Var[\psi(Z+X)]}{\partial Z}$ to the corresponding "left" eigenvector – equal to the the transposed eigenvector of the transposed matrix, $\frac{\partial E[\psi(Z+X)]}{\partial Z'}$ Z'. For a zero mean variable X, using (1.4):

$$Z \frac{\partial E[\psi(Z+X)]}{\partial Z} = Z\,[\frac{\partial \psi}{\partial Z} + \frac{1}{2} \frac{\partial tr\left(\frac{\partial^2 \psi}{\partial Z \partial Z'} V\right)}{\partial Z}] = Z[\frac{\partial \psi}{\partial Z} + \frac{1}{2} vec(V)' \frac{\partial \left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)}{\partial Z}]$$

(3.13)

$\frac{\partial Var[\psi(Z+X)]}{\partial Z}$ is given by (1.9). Transposing (3.12), denoting {Z $\frac{\partial E[\psi(Z+X)]}{\partial Z}$ - u π I_r}' = { $\frac{\partial E[\psi(Z+X)]}{\partial Z'}$ Z' - u π I_r} by A and $\frac{\partial Var[\psi(Z+X)]}{\partial Z'}$ by W, (3.13) has the form Y = A W = 0. Using Proposition A.5 of Appendix 1, we now require for any change in Z and vec(V) and/or π forming that:

$$\frac{\partial Y}{\partial \alpha} = (W' \otimes I_r) \frac{\partial A}{\partial \alpha} + A \frac{\partial W}{\partial \alpha} = 0 \quad (3.14)$$

The properties of the new solution turned out difficult to disentangle. An increase in π only will imply:

$$dvec(A') = dvec\{Z \frac{\partial \psi}{\partial Z} + \frac{1}{2} Z\, vec(V)' \frac{\partial \left(\frac{\partial^2 \psi}{\partial Z \partial Z'}\right)}{\partial Z}\}/dZ \;\; dZ \; - \; u\, vec(I_r)\, d\pi$$

A change in elements of V can be inspected through the implicit change in vec(V) at a fixed π. Developing the vector form of the left hand-side with Proposition A.1 of Appendix 1:

Using Proposition A.1.1, A.5 and D.2.1 in the Appendix 1:

$$\frac{\partial[vec(Z\frac{\partial\psi}{\partial Z})]}{\partial Z} = \frac{\partial[(I_r \otimes Z)\frac{\partial\psi}{\partial Z'}]}{\partial Z} = (\frac{\partial\psi}{\partial Z'} \otimes I_r) + (I_r \otimes Z)\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)$$

Using Proposition A.2 of Appendix 1 – vector of the product rule:

$$\text{vec}[\ Z\ \text{vec}(V)'\ \frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}] = \{I_r \otimes [Z\ \text{vec}(V)']\}\ \text{vec}[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}]$$

Through Proposition A.1, A.5 and D.2.1 in the Appendix 1:

$$\text{dvec}\{Z\text{vec}(V)'\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}\}/dZ = \{\text{vec}[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}]' \otimes I_{rr}\}\ [\text{vec}(I_r) \otimes \text{vec}(V) \otimes I_r] +$$

$$+ \{I_r \otimes [Z\ \text{vec}(V)']\}\ \frac{\partial\left[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}\right]}{\partial Z}$$

For V, an intermediate result is:

$$\text{dvec}\{Z\ \text{vec}(V)'\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}\}/\text{dvec}(V) = \{\text{vec}[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z'}]'\ [\text{vec}(I_r) \otimes I_{rr}]\} \otimes Z$$

We can confront this expression with that of the von Neumann-Morgenstern agent, implicit in (3.4). It has obvious similarities, but it is weighed by Z.

4. The Value of Ex-post Flexibility

4.1. The von Neumann Morgenstern Entity

Suppose the expected function maximizing agent can react – contingent on, point-wise - to X. Then, it sets Z such that:

$$\frac{\partial\psi(Z+X)}{\partial Z} = 0 \tag{4.1}$$

Then it will choose Z as a function of X such that:

$$Z = Z(X) = Y - X \tag{4.2}$$

where Y is the constant for which:

$$\frac{\partial \psi(Y)}{\partial Z} = 0 \tag{4.3}$$

It will always be the case, no matter what value X takes, that:

$$\psi(Z+X) = \psi(Y) \tag{4.4}$$

If $E[X] = \mu = 0$:

$$E[Z] = Y \quad ; \quad Var[\psi(Z+X)] = 0 \tag{4.5}$$

Obviously – see Martins (2004a), if the risks surround the decision variables:

Proposition 8: The flexible von Neumann-Morgenstern agent will:

1. balance any randomness X by a corresponding compensation in Z, rendering the objective function completely stable.

2. exhibit an expected policy $E[Z] = Y$ higher (lower) than the ex-ante committed agent iff $dZ / dvec(V) < (>) 0$ for the latter.

4.2. The Mean-Variance Agent

Consider a mean-variance unit. On the one hand, even if it cannot control Z, provided it can react after observing X, we can admit that it has the ability to throw away a "chunk", y, of $\psi(Z + X)$. Such ability is never used by an expected value maximizer, of course. But will by the current entity. It has now a series of decisions $y = y(X)$, a random variable the probability distribution of which will be in line with that of X.

Admit that $X \sim f(X)$, $a < X < b$. The entity will choose y's in such a way that it:

$$\underset{y}{Max}\ U\{E[\psi(Z+X) - y], Var[\psi(Z+X) - y]\} =$$

$$= U(E[\psi(Z+X) - y], E\{[\psi(Z+X) - y]^2\} - E[\psi(Z+X) - y]^2)$$

$$= U(\int_a^b [\psi(Z+X) - y]\ f(X)\ dX, \int_a^b [\psi(Z+X) - y]^2\ f(X)\ dX - \{\int_a^b [\psi(Z+X) - y]$$
$$f(X)\ dX\}^2) \tag{4.6}$$

$\int_a^b$ denotes r integral signs limited by the elements of vectors a and b, and dX stands for the product of the r differentials of the X's. A first thing to notice is the oddity of the problem: the controls are a continuum of values. But one can find variational problems in the theory of risk – see Karni (1979) assessing risk-sharing across states of nature[9]. The most unfamiliar feature is the dependency of the objective functional on expectations of functions of the control itself.

It is easily visualized through the development of the integrals that the optimal y's will be such that:

$$- U_1\{E[\psi(Z+X) - y], Var[\psi(Z+X) - y]\} f(X) - U_2\{E[\psi(Z+X) - y], Var[\psi(Z+X) - y]\}$$

$$\{2 [\psi(Z+X) - y] f(X) - 2 E[\psi(Z+X) - y] f(X)\} = 0 \quad (4.7)$$

y – or rather y(X), once they are conditional on X - will react to X according to:

$$y - E[y] = \psi(Z+X) - E[\psi(Z+X)] + \frac{1}{2}\frac{U_1\{E[\psi(Z+X)-y], Var[\psi(Z+X)-y]\}}{U_2\{E[\psi(Z+X)-y], Var[\psi(Z+X)-y]\}} \quad (4.8)$$

Then, taking expectations we conclude that the y's will be set in such a way to guarantee:

$$\frac{U_1\{E[\psi(Z+X)-y], Var[\psi(Z+X)-y]\}}{U_2\{E[\psi(Z+X)-y], Var[\psi(Z+X)-y]\}} = 0 \quad (4.9)$$

or

$$U_1\{E[\psi(Z+X)] - E[y], Var[\psi(Z+X) - y]\} = 0 \quad (4.10)$$

and, because $\frac{U_1\{E[\psi(Z+X)-y], Var[\psi(Z+X)-y]\}}{U_2\{E[\psi(Z+X)-y], Var[\psi(Z+X)-y]\}}$ is indeed constant, we can conclude from – squaring and taking expectations... - (4.8) that:

$$Var(y) = Var[\psi(Z+X)] = Cov[y, \psi(Z+X)] \quad (4.11)$$

insuring perfect correlation between y and $\psi(Z+X)$ - as expected - and:

$$Var[\psi(Z+X) - y] = 0 \quad (4.12)$$

E[y] will be such that:

$$U_1\{E[\psi(Z+X)] - E[y], 0\} = 0 \quad (4.13)$$

implying:

$$U_1\{E[\psi(Z+X)], 0\} - U_{11}\{E[\psi(Z+X)], 0\} E[y] + \frac{1}{2} U_{111}\{E[\psi(Z+X)], 0\} E[y]^2 + \ldots = 0$$

[9] Our argument is different from his, of course: we are assessing throwing away utility – not the argument of the function - after the random event occurs. As noted, the von-Neumann Morgenstern entity – that Karni overviews - would not accept to do it.

An approximation to the first order will require that optimally:

$$E[y] = \frac{U_1\{E[\psi(Z+X)],0\}}{U_{11}\{E[\psi(Z+X)],0\}} \tag{4.14}$$

As long as U(.,.) is convex in the first argument, E[y] > 0. (But then we might have a minimum with the policy – for a maximum, $U_{12}\{E[\psi(Z+X) - y], Var[\psi(Z+X) - y]\}$ must be sufficiently negative.)

We did not complicate the problem considering | y | subtracted from the function, or impose the restriction y > 0, using Khun-Tucker conditions - nor requiring E[y] > 0. Nevertheless, a negative y with E[y] > 0 may be accountingly meaningful: if the firm could interchange revenue allocation between periods, it would understate profits in good times, and overstate in bad times, transferring results in accordance to (4.8) - which implies that an optimal policy will render "net" utility, ψ(Z + X) – y, constant:

$$E[\psi(Z+X)] - E[y] = \psi(Z+X) - y \tag{4.15}$$

The agent will be willing to pay (loose) as much as g, the direct risk-premium of (2.20), for the possibility. Ideally, it will loose E[y] of expected ψ(Z + X) - of E[ψ(Z + X)] - for it. An expected value-maximizing entity would have no interest in engaging in such practices.

Proposition 9: A mean-variance agent that can react after the realization of the random event (even if not through Z, the exogenous deterministic variable):

1. may find it utility-yielding to "throw away" profits and even expected profits.

2. will choose the optimal dissipation to be increasing in the state of nature – in the observed ψ(Z + X).

3. may find desirable to accommodate through the policy all the randomness of ψ(Z + X).

Consider that Z can also be chosen by the agent. Then, it will solve a joint infinite series of conditional decisions in y and Z such that:

$$\underset{Z,y}{Max}\ U\{E[\psi(Z+X) - y], Var[\psi(Z+X) - y]\}$$

The F.O.C. with respect to y still hold. That will imply that the entity will use y to cushion all variability in "net" profits. If it does, it chooses Z such that:

$$\underset{Z}{Max}\ U\{E[\psi(Z+X) - y], 0\} \tag{4.16}$$

setting Z's such that:

$$\frac{\partial \psi(Z+X)}{\partial Z} = 0 \tag{4.17}$$

that is, it will mimic the behavior of a von Neumann-Morgenstern utility maximizer towards Z.

Consider that Z can be chosen by the agent but policy y is not meaningful:

$$\underset{Z}{Max} \;\; U\{E[\psi(Z+X)], Var[\psi(Z+X)]\} =$$

$$= U(E[\psi(Z+X)], E\{[\psi(Z+X)]^2\} - E[\psi(Z+X)]^2)$$

$$= U\{\int_a^b \psi(Z+X)\, f(X)\, dX, \int_a^b \psi(Z+X)^2\, f(X)\, dX - [\int_a^b [\psi(Z+X)\, f(X)\, dX]^2\}$$

It is easily visualized that the optimal Z's will obey:

$$[U_1\{E[\psi(Z+X)], Var[\psi(Z+X)]\} + U_2\{E[\psi(Z+X)], Var[\psi(Z+X)]\}$$

$$\{2\,\psi(Z+X) - 2\,E[\psi(Z+X)]\}]\; \frac{\partial \psi(Z+X)}{\partial Z}\; f(X) = 0 \tag{4.18}$$

Then Z will be set in such a way that either

$$\frac{\partial \psi(Z+X)}{\partial Z} = 0 \tag{4.19}$$

and Z is always equal to Y – X, where Y is the value for which $\frac{\partial \psi(Y)}{\partial Z} = 0$ – and the variance of $\psi(Z+X)$ is completely eliminated.
Or:

$$\psi(Z+X) = E[\psi(Z+X)] - \frac{1}{2}\,\frac{U_1\{E[\psi(Z+X)], Var[\psi(Z+X)]\}}{U_2\{E[\psi(Z+X)], Var[\psi(Z+X)]\}} \tag{4.20}$$

Again, the optimal Z's would make Z + X constant. Yet, taking expectations we conclude that for this solution to hold all over the domain, the Z's would be set in such a way to guarantee:

$$\frac{U_1\{E[\psi(Z+X)], Var[\psi(Z+X)]\}}{U_2\{E[\psi(Z+X)], Var[\psi(Z+X)]\}} = 0 \tag{4.21}$$

That will also require – replacing it in (4.20) - that Z will be such that:

$$\psi(Z+X) = E[\psi(Z+X)] \tag{4.22}$$

and, as

$$U_1\{E[\psi(Z+X)], Var[\psi(Z+X)]\} = U_1\{E[\psi(Z+X)], 0\} = 0 \tag{4.23}$$

we enter structure (4.16) again.

We conclude that the transfer is, in any case, completely accomplished if ex-post adjustability of the control variable to which the risk is added is available. Then, adjustability through y becomes redundant.

Proposition 10: A mean-variance agent that can react after the realization of the random and choose Z, the variable to which it is added to:

1. achieves the same solution as the expected-value maximizer.
2. Proposition 8 applies, comparisons valid with the von Neumann-Morgenstern ex-ante committed agent.
3. dispenses with other smoothing tools.

5. Mixed Environments: A Final Comment.

To reproduce particular environments, we may want to combine the three types of situations – that is, in Z = (Z1, Z2, Z3), there will be variables Z1, which the agent can but endure, others, Z2, that he can decide before the realization of the added risk, and others, Z3, that he can adjust after the randomness is observed.

The von Neumann-Morgenstern individual will:

$$\underset{Z_2,Z_3}{Max} \quad E[\psi(Z+X)] = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \psi(Z+X)\, f(X)\, dX \tag{5.1}$$

F.O.C are of two types: a unique one with respect to Z2:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \frac{\partial\psi(Z+X)}{\partial Z_2}\, f(X)\, dX = 0 \tag{5.2}$$

Infinite ones for Z3:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{\partial\psi(Z+X)}{\partial Z_3}\, f(X)\, dX_1\, dX_2 = 0 \tag{5.3}$$

From (5.3), a continuum of conditional optimal of policies are derived for Z3, function of X3, of the common Z2, and of the parameters of the joint distribution of X = (X1, X2, X3). It can then be replaced in (5.2) to solve for Z2. If the distribution of the vector X3 is independent of that of the vector (X1, X2) - i.e., if we can write f(X) = f(X1, X2, X3) = f12(X1, X2) f3(X3), where f12(X1, X2) and f3(X3) denote the marginal probability distributions -, (Z3 + X3) is a constant vector in the optimal policies and the randomness in that sum is always neutralized. Yet, that constant level will not be the one for which $\frac{\partial\psi(Z+X)}{\partial Z}$ = 0, unless $\frac{\partial\psi(Z+X)}{\partial Z_3}$ is invariant to (does not depend on) (Z1 + X1, Z2 + X2)...

Notice that if f(X) = f(X1, X2, X3) = f12(X1, X2) f3(X3), we can use the expansion of Proposition 1 applied only to (Z1, Z2), take the derivative with respect to Z3 and equate it to zero to approximate (5.3), but not otherwise.

For a mean-variance agent:

$$\underset{Z_2,Z_3}{Max}\ U\{E[\psi(Z+X)],\ Var[\psi(Z+X)]\} =$$

$$= U(E[\psi(Z+X)],\ E\{[\psi(Z+X)]^2\} - E[\psi(Z+X)]^2)$$

$$= U\{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \psi(Z+X)\ f(X)\ dX, \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \psi(Z+X)^2 f(X)dX - [\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} [\psi(Z+X)f(X)\ dX]^2\} \quad (5.4)$$

It is easily visualized that the optimal Z's will obey:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} [U_1\{E[\psi(Z+X)],\ Var[\psi(Z+X)]\} + U_2\{E[\psi(Z+X)],\ Var[\psi(Z+X)]\}$$

$$\{2\ \psi(Z+X)\ -\ 2\ E[\psi(Z+X)]\}]\ \frac{\partial\psi(Z+X)}{\partial Z_2}\ f(X)\ dX = 0 \quad (5.5)$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} [U_1\{E[\psi(Z+X)],\ Var[\psi(Z+X)]\} + U_2\{E[\psi(Z+X)],\ Var[\psi(Z+X)]\}$$

$$\{2\ \psi(Z+X)\ -\ 2\ E[\psi(Z+X)]\}]\ \frac{\partial\psi(Z+X)}{\partial Z_3}\ f(X)\ dX_1\ dX_2 = 0 \quad (5.6)$$

Expressions become more complicated, but constancy of ($Z3 + X3$) in case of statistical independence is preserved. It will, however, differ from that of a expected value maximizer. And that will still be true if no ex-ante control is available as long as some additive uncertainty surrounds out-of-decision range variables.

6. Production Theory Applications

We admit a firm that produces output, q, sold at price P and employing r inputs, of quantities Li , i=1,2,...,r, represented by a column vector L, at unit (column-vector) cost w, of element wi. Its technology is represented by a production function q = F(L), continuous, increasing, quasi-concave and differentiable to several orders in L.

Under certainty, it has a deterministic cost function C(q, w) continuous, increasing, concave and differentiable to several orders in q, a profit function π(P, w), both enjoying the usual properties (Varian, 1992) and compatible with technology F(L).

Uncertainty has been apposed to the firm's problem in several contexts (Oi, 1961; Sandmo, 1971; Feldstein, 1971; Rothemberg & Smith, 1971; Batra & Ullah, 1974) -Aiginger (1987) surveys several scenarios, and a recent univariate inquiry can be found in Martins (2007).

6.1. Price Uncertainty under Ex-post Flexibility

The firm acts towards prices optimizing the profits after observing the randomness. Obviously, the expected value maximizing firm will react to X according to $\psi(P + X1, w + Xr)$ that takes the role of $\psi(Z + X)$ and the conclusions of section 2 apply. $\psi(P + X1, w + Xr)$ is convex in (P, w) and general risk-loving behaviour towards the randomness – negative risk-premium – is expected. As for the particular problem, the convexity of the objective function is related to the magnitudes of the slopes of

- supply, once $\frac{\partial \Pi(P,w)}{\partial P}$ = qS(P, w)
- input derived demands, once - $\frac{\partial \Pi(P,w)}{\partial w}$ = $L^D(P, w)$

they will determine the size order of the impact of uncertainty on the maximand. Of course, the size of the impact of uncertainty on the expected supply and demand themselves is determined by their own concavity in the corresponding arguments – being negative when the functions are concave, positive when convex.

Notice, however, that the mean-variance firm – staying on the market long enough to experience the fluctuations of the profits - may not find it optimal to react according to $\pi(P + X1, w + Xr)$. The firm may trade expected profits by less volatile income. Then, it may enter into the scenario of section 4.2.: we conclude that a mean-variance entity with ex-post flexibility may find it optimal to engage in charitable contributions in good states. If the variability comes from the input prices, in which case it is likely that $Uw < 0$, and we consider a vector Y subtracted to X, it would be more likely that second order conditions will be satisfied with such a policy; then firms would be willing to pay higher employee compensations in good times, for example.

6.2. Quantity Uncertainty under Ex-ante Commitment

Under ex-ante commitment with respect to the control variables, the firms are in the environment of section 3 and $\sigma(Z)$ becomes P F(L) – w L. Uncertainty added to the control variables has the size of the effect on the maximand determined by that of the simple addition of the randomness, evaluated at the optimal control. It is determined by the concavity of the production function itself.

The firm equates the value of expected marginal product – the expected inverse factor demands - to factor prices:

$$P \frac{\partial E[F(L+X)]}{\partial L} = P \ E[\frac{\partial F(L+X)}{\partial L}] = w \qquad (6.1)$$

Then L will move in the same way as $\frac{\partial E[F(L+X)]}{\partial L}$ reacts to uncertainty. The more concave (less convex) the inverse demands – and potentially also demands, once they are negatively sloped - are[10] , the more $\frac{\partial E[F(L+X)]}{\partial L}$ decreases with uncertainty at a given level L. To compensate a rise in uncertainty – being inverse demands negatively sloped -, if the marginal product function is concave (convex), a lower (higher) level of the input will be sought.

[10] See Carroll & Kimball (1996) for an assessment of the role of the concavity of the inter-temporal consumption function under uncertainty.

Analogous lines would allow the interpretation of the effect of uncertainty in X affecting the cost function $C(q+X)$. Then:

- the impact on expected profits of a rise in uncertainty in q will be more negative the more convex is the cost function – the higher the slope of the marginal cost function, the lower the slope of output supply qS(P), its inverse function.

- as the firm sets:

$$P = \frac{\partial E[C(q+X)]}{\partial q} = E[\frac{\partial C(q+X)}{\partial q}] \quad (6.2)$$

The more concave (less convex) is the marginal cost function – the more convex is the supply, its inverse function, once it is positively sloped -, the higher will be the increase in q required to balance an increase in uncertainty. If marginal cost is convex (concave), the optimal q decreases (increases) with uncertainty.

7. Conclusion

Matrix representation of risk-premium and corresponding first differentials with respect to exogenous parameters of multivariate random variables was presented. They are useful to generate theoretical conclusions of several economic applications, but also to simulate empirically the effect of risk exposure in any environment, once functional forms are specified. More distantly, the principles used and developed in the text may reveal themselves useful for algorithms requiring numerical differentiation - potentially, with application in initial-value generation in non-linear optimization.

We concluded about the importance and role of third and higher order derivatives in the analysis of risk-aversion and decision-making under uncertain backgrounds. General features of both issues' crucial vectors diverge for an expected-value maximizer and a mean-variance one. In general, higher moments and derivatives (differentiation) are recommended for the latter to achieve the same order approximation of the results. Reliance on Taylor's expansion – common in the risk literature – also originated a straight-forward connection between the multivariate measure of the aversion in the attitude to multivariate risks and the (partial) aversion to each of the elementary risks subject to background uncertainty.

In general, and as intuitively expected, a mean-variance ("utility") entity potentially exhibits a "compound-premium", weighing the expected value but also the variance impact of an exogenous noise. Interestingly, if given the possibility of transferring utility across states of nature, a rational mean-variance agent with a sufficiently convex utility in the expected value argument, will approach the von Neumann-Morgenstern attitude.

Subject to uncertainty, whenever possible – with ex-post adjustment of control variables or by other means –, both types of agents will try to reach the maximum value of the function of the expected value of the (random or not) arguments. With enforcing contracts with respect to the controls, the expected optimal maximand reacts to uncertainty as the expected function would in the absence of optimization – but at the optimal level of the control variables.

With ex-post flexibility with respect to decision variables to which the risk is added, uncertainty is completely countervailed – and the optimized function completely stabilized.

Production applications under some of the relevant environments – as consumption could have also been – were briefly overviewed. The conditioning effect of concavity, slopes of supply and factor demand were appropriately related to the response to uncertainty by a competitive firm.

Appendix

Appendix 1.

. We use:

Convention 1. Let A be an m x n matrix the elements of which depend on the r element column vector α. Then $\frac{\partial A}{\partial \alpha}$ (a Jacobian matrix) is a mn x r matrix that has in the i-th row and j-th column element the derivative of i-th element of the vector vec(A) – created juxtaposing consecutively the n columns of A in a single "column" - with respect to the j-th element of vector α:

$$\frac{\partial A}{\partial \alpha} = \frac{\partial vec(A)}{\partial \alpha}$$

Convention 2. We will write $\frac{\partial A}{\partial \alpha'} = \left(\frac{\partial A}{\partial \alpha}\right)'$.

Convention 3. We will denote by $\frac{\partial^2 A}{\partial \alpha \partial \alpha'} = \frac{\partial \left[\left(\frac{\partial A}{\partial \alpha}\right)'\right]}{\partial \alpha} = \frac{\partial vec\left[\left(\frac{\partial A}{\partial \alpha}\right)'\right]}{\partial \alpha}$.

For example, if m = n = 1, $\frac{\partial A}{\partial \alpha} = [\frac{\partial A}{\partial \alpha_1} \; \frac{\partial A}{\partial \alpha_2} \; \ldots \; \frac{\partial A}{\partial \alpha_r}]$ and $\frac{\partial^2 A}{\partial \alpha \partial \alpha'}$ is the Hessian matrix of the function A, matrix with typical element $[\frac{\partial^2 A}{\partial \alpha_i \partial \alpha_j}]$. Being A a scalar, $\frac{\partial^2 A}{\partial \alpha \partial \alpha'} = \frac{\partial^2 A}{\partial \alpha' \partial \alpha}$; $\frac{\partial^2 A}{\partial \alpha' \partial \alpha_j} = [\frac{\partial^2 A}{\partial \alpha_1 \partial \alpha_j} \; \frac{\partial^2 A}{\partial \alpha_2 \partial \alpha_j} \; \ldots \; \frac{\partial^2 A}{\partial \alpha_r \partial \alpha_j}]'$.

We refer below useful propositions on matrix algebra used in the text. I_j denotes an identity j x j matrix.

. Quoting from Dhrymes (1978), often used results:

Proposition A.1. (Dhrymes, 1978, Proposition 86, p. 519). Let A be m x n and B n x s. Then:

1. $vec(A\,B) = (I_s \otimes A)\, vec(B) = (B' \otimes I_m)\, vec(A)$

(Hence:) 2. $vec(A) = (I_n \otimes A)\, vec(I_n) = (A' \otimes I_m)\, vec(I_m)$

Proposition A.2. (Dhrymes, 1978, Corollary 22, p. 519).

$$vec(A_1 A_2 A_3) = [I \otimes (A_1 A_2)]\, vec(A_3)$$

Proposition A.3. (Dhrymes, 1978, Proposition 88, p. 521).

$$tr(A\,B) = vec(A')'\, vec(B) = vec(B')'\, vec(A)$$

Proposition A.4. (Dhrymes, 1978, Remark 45, p. 522).

$$tr(A_1 A_2 A_3 A_4) = vec(A_2')'\,(A_1' \otimes A_3)\, vec(A_4) = vec(A_4')'\,(A_3' \otimes A_1)\, vec(A_2)$$

Proposition A.5. (Dhrymes, 1978, Proposition 93, p. 525). Let Y = A X, where Y is m x 1, A is m x n, X is n x 1, with both A and X dependent on the vector α, r x 1.

$$\frac{\partial Y}{\partial \alpha} = (X' \otimes I_m)\frac{\partial A}{\partial \alpha} + A\frac{\partial X}{\partial \alpha}$$

Proposition A.6. (Dhrymes, 1978, Proposition 96, p. 527) Let Z be mx1, A mxn and X nx1, A is independent of the rx1 vector α. Then

$$\frac{\partial (Z'AX)}{\partial \alpha} = X'A'\frac{\partial Z}{\partial \alpha} + Z'A\frac{\partial X}{\partial \alpha}$$ and

$$\frac{\partial^2 (Z'AX)}{\partial \alpha \partial \alpha'} = \left(\frac{\partial Z}{\partial \alpha}\right)' A \frac{\partial X}{\partial \alpha} + \left(\frac{\partial X}{\partial \alpha}\right)' A' \frac{\partial Z}{\partial \alpha} + (X'A' \otimes I_r)\frac{\partial^2 Z}{\partial \alpha \partial \alpha'} + (Z'A \otimes I_r)\frac{\partial^2 X}{\partial \alpha \partial \alpha'}$$

Proposition A.7. (Dhrymes, 1978, Proposition 100, p. 532) Let A be mxn, X nxq, B qxr and Z rxm. If X and Z are functions of the rx1 vector α. Then

$$\frac{\partial tr(AXBZ)}{\partial \alpha} = vec(A'Z'B')'\frac{\partial vec(X)}{\partial \alpha} + vec(B'X'A')'\frac{\partial vec(Z)}{\partial \alpha}$$

Proposition A.8. (Dhrymes, 1978, Proposition 101, p. 532) Let A and B are square matrices m x m and q x q respectively, and only X – which is qxm - depends on the rx1 vector α. Then

$$\frac{\partial tr(AX'BX)}{\partial \alpha} = vec(X)'\,[(A' \otimes B) + (A \otimes B')]\frac{\partial vec(X)}{\partial \alpha}$$

. Others:

Proposition B.1. Being A an (mxn) matrix (see the result in Hamilton, 1994, p. 733):
1. $I_r \otimes (I_s \otimes A) = I_{rs} \otimes A$
2. $(A \otimes I_r) \otimes I_s = A \otimes I_{rs}$
Proof: Use the fact that, for any matrices A, B and C, $A \otimes B \otimes C = A \otimes (B \otimes C)$.

Proposition B.2. Being X an nx1 column vector and Z an rx1 one:
1. $X \otimes Z' = XZ'$
2. $X' \otimes Z = (XZ')' = ZX' = Z \otimes X'$
3. $X \otimes Z' = Z' \otimes X$
4. $X \otimes X' = X' \otimes X = XX'$
5. $vec(X \otimes Z') = (Z \otimes I_n)X = (I_r \otimes X)Z = Z \otimes X$ (Proof: Use A.1.1.)

Proposition B.3. Being X a column vector: (Proofs: Use B.2.4.)
1. $(XX') \otimes X = X \otimes (XX')$
2. $[X \otimes (XX')]' = X' \otimes (XX') = (XX') \otimes X'$

Proposition B.4. Being X a column vector: (Proofs: Use B.2.4. and B.2.5.)
1. $X \otimes X = vec(XX')$
2. $X \otimes X \otimes X = vec[(XX') \otimes X] = vec[X' \otimes (XX')]$
3. $X \otimes X \otimes X \otimes X = vec[(XX') \otimes (XX')]$

Proposition C.1. Let X be an nx1 vector and A a pxs matrix. Then,
1. $vec(A \otimes X) = vec(A) \otimes X$
2. $vec(X \otimes A) = vec(A \otimes X') = [(I_s \otimes X) \otimes A]\, vec(I_s) = (X' \otimes A' \otimes I_{np})\, vec(I_{np})$
3. $vec(I_s \otimes X) = vec(I_s) \otimes X$
4. $vec(X \otimes I_s) = vec(I_s \otimes X') = [(I_s \otimes X) \otimes I_s]\, vec(I_s) = (X' \otimes I_{sns})\, vec(I_{ns})$
(Proofs: Use A.1.2.)

Proposition C.2. Let A be an mxn matrix and B an rxs one - $(A \otimes B)$ is mr x ns. Then, $vec(A \otimes B) =$
1. $[I_{ns} \otimes (A \otimes I_r)]\, vec(I_n \otimes B)$
2. $[(I_n \otimes B') \otimes I_{mr}]\, vec(A \otimes I_r)$
3. $[I_{ns} \otimes (I_m \otimes B)]\, vec(A \otimes I_s) = (I_{nsm} \otimes B)\, vec(A \otimes I_s)$
4. $[(A' \otimes I_s) \otimes I_{mr}]\, vec(I_m \otimes B) = (A' \otimes I_{smr})\, vec(I_m \otimes B)$

Proof: Use the fact that $(A \otimes B) = (A \otimes I_r)(I_n \otimes B) = (I_m \otimes B)(A \otimes I_s)$ and Proposition A.1.1.

Proposition D.1. Let a be a scalar and B an (mxn) matrix, both functions of the elements of an (rx1) vector α. Then:

$$\frac{\partial(aB)}{\partial\alpha} = \frac{\partial vec(aB)}{\partial\alpha} = vec(B)\frac{\partial a}{\partial\alpha} + a\frac{\partial B}{\partial\alpha}$$

Proposition D.2. Being X an (nx1) vector dependent on a (rx1) vector α and A a pxs matrix independent of α:

1. $$\frac{\partial(A\otimes X)}{\partial\alpha} = \frac{\partial vec(A\otimes X)}{\partial\alpha} = vec(A)\otimes\frac{\partial X}{\partial\alpha}$$ (Proof: Obvious from C.1.1.)

2. $$\frac{\partial(X\otimes A)}{\partial\alpha} = \frac{\partial vec(X\otimes A)}{\partial\alpha} = \{[vec(\frac{\partial X}{\partial\alpha})]'\otimes A'\otimes I_{np}\}[I_r\otimes vec(I_{np})]$$

3. $$\frac{\partial(X\otimes I_s)}{\partial\alpha} = \frac{\partial vec(X\otimes I_s)}{\partial\alpha} = \{[vec(\frac{\partial X}{\partial\alpha})]'\otimes I_{sns}\}[I_r\otimes vec(I_{ns})]$$

4. $$\frac{\partial(\alpha\otimes\alpha)}{\partial\alpha} = I_r\otimes\alpha + \alpha\otimes I_r$$

Appendix 2.

. Taylor's expansion to the fourth order of any function ψ(Z) around neighbourhood X of a given level Z generates (see an approximation to the third order in sum notation in Hamilton 1994, p. 738):

(A.1)
$$\psi(Z+X) = \psi(Z) + \frac{\partial\psi}{\partial Z}X + \frac{1}{2!}\left[vec\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)\right]'(X\otimes X) +$$

$$+\frac{1}{3!}\left\{vec\left[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}\right]\right\}'(X\otimes X\otimes X) +$$

$$+\frac{1}{4!}\left(vec\left\{\frac{\partial\left[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}\right]}{\partial Z}\right\}\right)'(X\otimes X\otimes X\otimes X) + \ldots =$$

$$= \psi(Z) + \frac{\partial\psi}{\partial Z}X + \frac{1}{2!}X'\frac{\partial^2\psi}{\partial Z\partial Z'}X +$$

$$+ \frac{1}{3!} \text{vec}(XX')' \frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z} X + \frac{1}{4!} \text{vec}[(XX') \otimes X]' \frac{\partial\left[\frac{\partial\left(\frac{\partial^2\psi}{\partial Z\partial Z'}\right)}{\partial Z}\right]}{\partial Z} X + \ldots$$

Proposition E. Let X be an rx1 random vector for which E[X] = μ and Cov(X) = E[(X - μ) (X - μ)'] = V. Then:

1. E[XX'] = V + μμ'
2. E{[(X - μ) (X - μ)'] ⊗ (X - μ)} = E{(X X'- Xμ' – μX' + μμ') ⊗ (X - μ)} =
   = E{[(X - μ) (X - μ)'] ⊗ X} – (V ⊗ μ) =
   = E[(X X'- Xμ' – μX' + μμ') ⊗ X] – (V ⊗ μ) =
   = E[(X X'- Xμ' – μX') ⊗ X] – [(V - μμ') ⊗ μ] =
   = E[(X X)' ⊗ X] - E[(Xμ') ⊗ X] - E[(μX') ⊗ X] – [(V - μμ') ⊗ μ]
   = E[(X X)' ⊗ X] - [μ' ⊗ vec(V+μμ')] – (μ ⊗ V) – (V ⊗ μ)
   = E[(X X)' ⊗ X] - [vec(V+μμ') ⊗ μ'] – (μ ⊗ V) – (V ⊗ μ)
   = E[(X X)' ⊗ X] - vec(V+μμ') μ' – (μ ⊗ V) – (V ⊗ μ)
3. E{[XX' – E(XX')] ⊗ (X -μ)} = E[(XX') ⊗ (X -μ)] = E[(XX') ⊗ X] – [(V + μμ') ⊗μ]
4. E{(X -μ) ⊗ [XX' - E(XX')]} = E[(X -μ) ⊗ (XX')] = E[X ⊗ (XX')] – [μ ⊗(V + μμ')]
5. E{[XX' – E(XX')]⊗(X-μ)'} = E[(XX')⊗(X-μ)'] = E[(XX') ⊗ X'] – [(V + μμ') ⊗μ']
6. E{(X-μ)'⊗ [XX' - E(XX')]} = E[(X-μ)'⊗ (XX')] = E[X' ⊗ (XX')] – [μ' ⊗(V + μμ')]
7. E{[XX' – E(XX')] ⊗ [XX' – E(XX')]} = E{(XX') ⊗ [XX' – E(XX')]} =
   = E[(XX') ⊗ (XX')] – [(V + μμ') ⊗ (V + μμ')]
8. E{[(X - μ) (X - μ)'] ⊗ (X - μ) (X - μ)'} = E{(X X'- Xμ' – μX' + μμ') ⊗ (X - μ) (X - μ)'} = E{(X X'- Xμ' – μX') ⊗ (X - μ) (X - μ)'} + (μμ') ⊗ V
   = E{[(X (X - μ)'] ⊗ (X - μ) (X - μ)'} - μ ⊗ E{(X - μ)' ⊗ [(X - μ) (X - μ)']} =
   = E{(X X') ⊗ [(X - μ) (X - μ)'}- E[(X μ') ⊗ [(X - μ) (X - μ)']} - μ ⊗ E{(X - μ)' ⊗ [(X -μ) (X - μ)']} = E[(X X') ⊗ (X X')] - E[(X X')⊗(μX')] - E[(XX') ⊗ (Xμ')] + [(V+μμ')⊗(μμ')] –
   - E[(X μ') ⊗ (XX')] + E[(X μ') ⊗ (μX')] + E[(X μ') ⊗ (Xμ')] - [(μ μ') ⊗ (μμ')] -
   - μ ⊗ E{(X - μ) ⊗ [(X - μ) (X - μ)']}' =
   = E[(X X') ⊗ (X X')] - E[(X X') ⊗ X'] ⊗ μ - E[(XX') ⊗ X] ⊗ μ' + [V ⊗ (μμ')] –
   - μ' ⊗ E[X ⊗ (XX')] + μ' ⊗ (V+μμ') ⊗ μ + μ' ⊗ vec(V+μμ') ⊗ μ' -
   - μ ⊗ { E[(X X)' ⊗ X] - [μ' ⊗ vec(V+μμ')] - (μ⊗ V) – (V ⊗ μ) }' =
   = E[(X X') ⊗ (X X')] - E[(X X') ⊗ X'] ⊗ μ - E[(XX') ⊗ X] ⊗ μ' + [V ⊗ (μμ')] –
   - μ' ⊗ E[X ⊗ (XX')] + μ' ⊗ (V+μμ') ⊗ μ + μ' ⊗ vec(V+μμ') ⊗ μ' -
   - μ ⊗ { E[(X X)' ⊗ X'] - [μ ⊗ vec(V+μμ')'] - (μ' ⊗ V) – (V ⊗ μ') } =
   = E[(X X') ⊗ (X X')] - E[(X X') ⊗ X'] ⊗ μ - E[(XX') ⊗ X] ⊗ μ' + [V ⊗ (μμ')] –
   - μ' ⊗ E[X ⊗ (XX')] + μ' ⊗ (V+μμ') ⊗ μ + μ' ⊗ vec(V+μμ') ⊗ μ' -
   - μ ⊗ { E[(X X)' ⊗ X'] - [vec(V+μμ')' ⊗ μ]- (μ' ⊗ V) – (V ⊗ μ') }

Appendix 3.

. Consider that X is a nx1 vector with multivariate normal distribution with E[X] = μ and Cov(X) = V. As is well known, denoting t by the (nx1) vector of arguments, its moment generating function is:

$$M(t) = \exp(\mu' t + \frac{t'Vt}{2})$$

Proposition F. Then:

1. $\frac{\partial M(t)}{\partial t'} = \exp(\mu' t + \frac{t'Vt}{2}) (\mu + V t)$
2. $\frac{\partial^2 M(t)}{\partial t'\partial t} = \exp(\mu' t + \frac{t'Vt}{2}) [(\mu + V t) (\mu + V t)' + V]$
3. $\frac{\partial\left[\frac{\partial^2 M(t)}{\partial t'\partial t}\right]}{\partial t} = \exp(\mu' t + \frac{t'Vt}{2}) (\{[I_n \otimes (\mu + V t)] (\mu + V t) + vec(V)\} (\mu + V t)' +$
   $+ [(\mu + V t)' \otimes I_{nn}] [vec(I_n) \otimes V] + [I_n \otimes (\mu + V t)] V) =$

$$= \exp(\mu' t + \frac{t'Vt}{2}) (\{[(\mu + Vt) \otimes (\mu + Vt)] + vec(V)\} (\mu + Vt)' + $$
$$+ [(\mu + Vt) \otimes V] + [V \otimes (\mu + Vt)])$$

$$= \exp(\mu' t + \frac{t'Vt}{2}) (\{vec[(\mu + Vt)(\mu + Vt)'] + vec(V)\} (\mu + Vt)' + $$
$$+ [(\mu + Vt) \otimes V] + [V \otimes (\mu + Vt)])$$

Proof: $vec(\frac{\partial^2 M(t)}{\partial t' \partial t}) = \exp(\mu' t + \frac{t'Vt}{2}) \{[I_n \otimes (\mu + Vt)] (\mu + Vt) + vec(V)\}$. Then, apply rule of differentiation of Propositions A.5 and D.2.1. and use B.2.5.

Note that $[(\mu + Vt)' \otimes I_{nn}] [vec(I_n) \otimes V] = \{[(\mu + Vt)' \otimes I_n] vec(I_n)\} \otimes V = (\mu + Vt) \otimes V$. (Use A.1.2.)

4. $$\frac{\partial \left\{ \frac{\partial \left[\frac{\partial^2 M(t)}{\partial t' \partial t} \right]}{\partial t'} \right\}}{\partial t} = \exp(\mu' t + \frac{t'Vt}{2}) \{ \{(\mu + Vt) \otimes vec[(\mu + Vt)(\mu + Vt)' + V]\} + $$
$$+ vec[(\mu + Vt) \otimes V] + vec[V \otimes (\mu + Vt)]\} (\mu + Vt)' + $$
$$+ \exp(\mu' t + \frac{t'Vt}{2}) (\{V \otimes vec[(\mu + Vt)(\mu + Vt)' + V]\} + $$
$$+ [(\mu + Vt)' \otimes I_{nnn}] \{vec(I_n) \otimes \frac{\partial vec[(\mu + Vt) \otimes (\mu + Vt)']}{\partial t}\} + $$
$$+ (V \otimes I_{nn}) \frac{\partial vec[(Vt) \otimes I_n]}{\partial t} + [vec(V) \otimes V]) = $$

$$= \exp(\mu' t + \frac{t'Vt}{2}) \{ \{(\mu + Vt) \otimes vec[(\mu + Vt)(\mu + Vt)' + V]\} + $$
$$+ vec[(\mu + Vt) \otimes V] + vec[V \otimes (\mu + Vt)]\} (\mu + Vt)' + $$
$$+ \exp(\mu' t + \frac{t'Vt}{2}) (\{V \otimes vec[(\mu + Vt)(\mu + Vt)' + V]\} + $$
$$+ [(\mu + Vt)' \otimes I_{nnn}] [vec(I_n) \otimes \{[V \otimes (\mu + Vt)] + [(\mu + Vt)' \otimes I_{nn}] [vec(I_n) \otimes V]\}] + $$
$$+ (I_{nn} \otimes V) [vec(V)' \otimes I_{nnn}] [I_n \otimes vec(I_{nn})] + [vec(V) \otimes V])$$

Proof: $$vec(\frac{\partial \left[\frac{\partial^2 M(t)}{\partial t' \partial t} \right]}{\partial t}) = \exp(\mu' t + \frac{t'Vt}{2}) \{\{I_n \otimes vec[(\mu + Vt)(\mu + Vt)' + V]\}(\mu + Vt) + $$
$$+ vec[(\mu + Vt) \otimes V] + vec[V \otimes (\mu + Vt)]\} = $$

$$= \exp(\mu' t + \frac{t'Vt}{2}) \{[(\mu + Vt)' \otimes I_{nn}] \; vec[(\mu + Vt)(\mu + Vt)' + V] + $$
$$+ vec[(\mu + Vt) \otimes V] + vec[V \otimes (\mu + Vt)]\} = $$

$$= \exp(\mu' t + \frac{t'Vt}{2}) \{ \{(\mu + Vt) \otimes vec[(\mu + Vt)(\mu + Vt)' + V]\} + $$
$$+ vec[(\mu + Vt) \otimes V] + vec[V \otimes (\mu + Vt)]\}$$

Using Proposition C.2 and A.5.

Proposition G. We will have that:
1. $M(0) = 1$.
2. $\frac{\partial M(0)}{\partial t'} = E[X] = \mu$.

3. $\frac{\partial^2 M(0)}{\partial t' \partial t}$ = E[X X'] = μμ' + V.

4. $\frac{\partial \left[\frac{\partial^2 M(0)}{\partial t' \partial t} \right]}{\partial t}$ = E[vec(X X') ⊗ X'] = E[(X X') ⊗ X] = E[X ⊗ (X X')] =

= $[(I_n \otimes \mu)\, \mu + vec(V)]\, \mu' + (\mu' \otimes I_{nn})\, [vec(I_n) \otimes V] + (I_n \otimes \mu)\, V$ =

= $[(\mu \otimes \mu) + vec(V)]\, \mu' + [(\mu' \otimes I_n) \otimes I_n]\, [vec(I_n) \otimes V] + (V \otimes \mu)$ =

= $[(\mu \otimes \mu) + vec(V)]\, \mu' + (\mu \otimes V) + (V \otimes \mu)$ =

= $vec(\mu\mu' + V)\, \mu' + (\mu \otimes V) + (V \otimes \mu)$

If μ = 0, E[(X X') ⊗ X] = 0. Hence, for the multivariate normal, E{[(X-μ)(X-μ)'] ⊗ (X-μ)} = 0 always.

5. $\frac{\partial \left\{ \frac{\partial \left[\frac{\partial^2 M(0)}{\partial t' \partial t} \right]}{\partial t'} \right\}}{\partial t}$ = E{vec[(X X') ⊗ X] ⊗ X'} =

{[μ ⊗ vec(μμ' + V)] + vec(μ ⊗ V) + vec(V ⊗ μ)} μ +

+ [V ⊗ vec(μμ' + V)] +

+ $(\mu' \otimes I_{nnn})\, [vec(I_n) \otimes \{(V \otimes \mu) + (\mu' \otimes I_{nn})\, [vec(I_n) \otimes V]\}]$ +

+ $(I_{nn} \otimes V)\, [vec(V)' \otimes I_{nnn}]\, [I_n \otimes vec(I_{nn})]$ + [vec(V) ⊗ V]

If μ = 0, E{vec[(XX') ⊗ X] ⊗ X'} = [V⊗vec(V)] + $(I_{nn} \otimes V)$ $\frac{\partial vec[(Vt) \otimes I_n]}{\partial t}$ (at t=0) + [vec(V)⊗V] = [V⊗vec(V)] + $(I_{nn} \otimes V)\, [vec(V)' \otimes I_{nnn}]\, [I_n \otimes vec(I_{nn})]$ + [vec(V) ⊗ V]

. It is easy to use the expressions to show that for a null means normal: $E[X_1 X_2 X_3] = 0$; $E[X_1 X_2 X_3 X_4] = \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{23}\sigma_{14}$ – see, for example, Dhrymes, 1978, p. 371 -, where σ_{ij} is the element of the i-th row j-th column of the symmetric matrix V.

References

Aiginger, K. (1987). *Production and Decision Theory under Uncertainty*. Oxford: Basil Blackwell.

Arrow, K.J. (1965). *Some Aspects of the Theory of Risk-Bearing*. Yrjo Jahnssonin Saatio, Helsinki.

Batra, R.N., & Ullah, A. (1974). Competitive firm and the theory of input demand under price uncertainty. *Journal of Political Economy*, 82(3), 537-548. doi. 10.1086/260211

Black, F. (1972). Capital market equilibrium with restricted borrowing. *Journal of Business*, 45(3), 444-455. doi. 10.1086/295472

Carroll, C.D., & Kimball, M.S. (1996). On the concavity of the consumption function. *Econometrica*, 64(4), 981-992.

Dhrymes, P.J. (1978). *Introductory Econometrics*. Springer-Verlag New York Inc.

Drèze, J.H. (1987). *Essays on Economic Decisions under Uncertainty*. Cambridge: Cambridge University Press.

Duffie, D. (1991). The theory of value in security markets. *In* W. Hildenbrand & H. Sonnenschein (Eds), *Handbook of Mathematical Economics. Vol. IV*. Amsterdam: North-Holland.

Duncan, G.T. (1977). A matrix measure of multivariate risk aversion. *Econometrica*, 45(4), 895-903. doi. 10.2307/1912680

Eichner, T., & Wagener, A. (2003). Variance vulnerability, background risks, and mean-variance preferences. *The Geneva Papers on Risk and Insurance Theory*. 28, 173-184. doi. 10.1023/A:1026396922206

Feldstein, M.S. (1971). Production with uncertain technology. *International Economic Review*, 12, 27-38. doi. 10.2307/2525494

Gollier, C. (2001). *The Economics of Risk and Time*. Cambridge: Massachusetts Institute of Technology.

Gollier, C.., & Pratt, J.W. (1996). Risk vulnerability and the tempering effect of background risk. *Econometrica*, 64(5), 1109-1123. doi. 10.2307/2171958

Hamilton, J.D. (1994). *Time Series Analysis*. Princeton: Princeton University Press.

Hartman, R. (1975). Competitive firm and the theory of input demand under price uncertainty: Comment. *Journal of Political Economy*, 83(6), 1289-1290. doi. 10.1086/260399

Hartman, R. (1976). Factor demand with output price uncertainty. *American Economic Review*, 66(4), 675-681.

Hirshleifer, J., & Riley, J.G. (1992). *The Analytics of Uncertainty and Information*. Cambridge: Cambridge University Press.

Karni, R.L. (1979). On multivariate risk aversion. *Econometrica*, 47(6), 1391-1401. doi. 10.2307/1914007

Karni, E., & Schmeidler, D. (1991). Utility theory with uncertainty. *In* W. Hildenbrand & H. Sonnenschein (Eds), *Handbook of Mathematical Economics. Vol. 4*. Amsterdam: North-Holland.

Keeney, R.L. (1973). Risk independence and multiattributed utility functions. *Econometrica*, 41(1), 27-34. doi. 10.2307/1913880

Kihlstrom, R., Romer, D., & Williams, S. (1981). Risk aversion with random initial wealth. *Econometrica*, 49, 911-920. doi. 10.2307/1912510

Kimball, M.S. (1990). Precautionary saving in the small and in the large. *Econometrica*, 58, 53-73. doi. 10.2307/2938334

Laffont, J.-J. (1989). *The Economics of Uncertainty and Information*. Cambridge: The Massachusetts Institute of Technology.

Lajeri-Chaherli, F. (2002). More on properness: The case of mean-variance preferences. *The Geneva Papers on Risk and Insurance Theory*, 27, 49-60. doi. 10.1023/A:1020681408308

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics*, 47(1), 13-37. doi. 10.2307/1924119

Markowitz, H. (1959). *Portfolio Selection. Cowles Foundation Monograph N. 16*. New York: John Wiley & Sons, Inc.

Martins, A.P. (2007). Uninsurable risks: Uncertainty in production, the value of information and price dispersion. *Annals of Economics and Finance*, 8, 341-383.

Martins, A.P. (2004). Attitude towards one risk, multiattributed utility, and the presence of correlated uncertainty. *In* Modelling and Analysis of Safety and Risk in Complex Systems: Proceedings of the Fourth International Scientific School MA SR – 2004, Saint-Petersburg, Russia, June 22-25, 2004) / SUAI. 2004: p. 30-53.

Oi, W.Y. (1961). The desirability of price instability under perfect competition. *Econometrica*, 29, 58-64. doi. 10.2307/1907687

Ormiston, M.B., & Schlee, E.E. (2001). Mean-variance preferences and investor behaviour. *The Economic Journal*, 111(474), 849-861. doi. 10.1111/1468-0297.00662

Pratt, J.W. (1964). Risk aversion in the small and in the large. *Econometrica*, 32, 122-136. doi. 10.2307/1913738

Rothemberg, T.J., & Smith, K.R. (1971). The effect of uncertainty on resource allocation. *Quarterly Journal of Economics*, 85, 440-453. doi. 10.2307/1885932

Sandmo, A. (1971). On the theory of the competitive firm under price uncertainty. *American Economic Review*, 61(1), 65-73.

Sharpe, W. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, 19, 425-442. doi.10.1111/j.1540-6261.1964.tb02865.x

Starmer, C. (2000). Developments in non-expected utility theory. *Journal of Economic Literature*, 38(2), 332-382. doi. 10.1257/jel.38.2.332

Tobin, J. (1958). Liquidity preference as behavior towards risk. *Review of Economic Studies*. 25(2), 68-85. doi. 10.2307/2296205

Varian, H.R. (1992). *Microeconomic Analysis*. 3rd Edition. New York: Norton.