Turkish Economic Review
www.kspjournals.org
Volume 3 March 2016 Issue 1

Wealth-In-Utility and Time-Consistent Growth: Real Excursions with an "Overlapping" Welfare Function

By Ana Paula MARTINS †

Abstract. This research explores the dynamic potential of point-wise utility functions optimization of representative agent economies. Such functions were generically considered to depend upon current consumption and wealth to be made available for next period usage or income generation, implying an endogenous (pseudo-)rate of time preference. At first inspection, the framework reproduced closely the dynamics and steady-state properties of the traditional Solow-Swan and Ramsey models – with population growth, exogenous technical progress, land, or increasing returns to scale - as well as, when human capital/knowledge was introduced, the Lucas-Uzawa endogenous growth set-up. General uncertainty – simulated at different decision stages - resulted in intuitively appealing solutions. Overlapping optimization of the capital stock generated forward-looking recursive dynamics. Homothetic preferences (CES or generalized Cobb-Douglas) - implying a constant consumption-(lead)wealth ratio along an optimal path and resulting in steady-state saving rates independent of CRS technologies features in simple structures -, were assumed for illustration, and also generic separable forms in the arguments. The latter were useful under uncertainty, allowing the inspection of the role of risk-aversion and diminishing marginal returns to capital in equilibrium and steady-state determination.
Keywords. Wealth-in-Utility (WIU); Capital-in-Utility (KIU). Overlapping Utility Functions. Consumption; Saving. Growth. Time (In)Consistency. Discounting; Rate of Time Preference. Growth under Uncertainty.
JEL. D91; D11. E19. O40.

1. Introduction

Maximization of an inter temporal utility function, usually with the form of accumulated discounted felicity (per period utility), has become a common objective of the representative agent in most macroeconomic and growth models. The formulation, after Ramsey (1928)[i], has proved successful in generating long-run and specially short-run and cyclical insights in the most varied economic subjects; however, it has the disadvantage of generating time inconsistent results. It is the purpose of this research to propose an alternative modelling framework capable of circumvent such shortcoming: assume that individuals proceed to the point-wise (or per period, in discrete time) maximization of an - eventually time indexed – utility function, with two types of arguments: perishable items (consumption, leisure), and assets. It is through the latter – a self

† Universidade Católica Portuguesa, Department of Economics,.This research started while the author was Invited Professor at Faculdade de Economia da Universidade Nova de Lisboa, Portugal.
☎. (351) 217 21 42 48
✉. apm@ucp.pt

as general bequest carried over to the next period - that concerns over future consumption are internalized.

On a superficial appraisal, the formulation would remind Sidrauski's (1967) money-in-utility functions. Or spirit-of capitalism-ones[ii]. In fact, it is quite different on both the rationale as on the dynamic implications: in MIU models, money is imbedded in felicity functions – themselves measures of per period flows of individuals' well-being -, and its inclusion meant to represent its favorable role in transactions. In spirit of capitalism models, wealth is included in the felicity function of intertemporal utility and its inclusion has been able to account for the savings puzzle[iii] and for the high volatility of stock prices[iv]. Our purpose is different: wealth (real wealth…) captures all intertemporal welfare substitution pertaining to a decision period (horizon) – felicity discounting is no longer required or justified. One can say that the traditional growth models as Solow (1956) and Swan (1956), assuming a constant savings function, are, to some extent, an inspiration of such a device; of course, asset demand becomes clearer, more immediate, with the proposed objective function. Moreover, first-order conditions confer a permanent income or life cycle flavor to the optimal consumption path.

In fact, intertemporal dynamic effects are quite present: even if individuals maximize static utility functions, they are (still…) conditioned by existing past wealth – they prepare today the wealth stock that will be available next period. On the other hand, they must form expectations to correctly assess today's real value of their possessions – or in a general equilibrium framework, those expectations will concur to generate their actual value. If we introduce leisure, future work-consumption-wealth decisions will also affect today's value of what we can call "full-time wealth". Moreover, a wealth plan for two periods may be evaluated today (a conditional decision over capital made in the past "overlapping" with the one made in the present) – and forward dynamics arise, with the same potential as future expectations models.

Overlapping generations[v] can also be simulated through the aggregation of coexisting cohorts supplies and demands – with younger generations exhibiting a stronger preference for capital relative to older ones. Technically, with the proposed function it becomes a matter of heterogeneity of contemporaneous agents.

In this article, we concentrate on the study of the potential of the modelling device – and work with discrete variables, even if continuous-time generalizations are straight-forward. Hence, we set out to replicate the dynamics of some of the widely recognized neoclassical benchmark environments, but under the new representative agent's behavior: firstly, the basic Solow-Swan and Ramsey-Cass-Koopmans one-sector models with their multiple extensions. Secondly, the endogenous growth Lucas-Uzawa experiment. Finally, we digress over the mechanisms behind exogenous shocks and their volatility transmission – or heterogeneity… - to economic aggregates under the current framework.

The exposition proceeds as follows: section 1 introduces the utility function and the representative agent dynamic problem; short-run dynamics and steady-state properties are explored in section 2, with the supportive market equilibrium briefly justified in section 3. Exogenous technical progress and the Lucas-Uzawa hypothesis concerning human capital formation are analyzed in section 4, implications of increasing returns to scale and fixed resources (land) studied in section 5. Section 6 deals with the effects of exogenous uncertainty, experimenting with both additive and multiplicative shocks. In section 7, an enlarged utility function, including lead capital as well, originates a recursive solution. A final appraisal and possible extensions produce a concluding section.

2. The Wealth-In-Utility Welfare Function

We will assume a generic utility function:

$$U_t(c_t, w_t) \quad (1)$$

c_t denotes (per capita) consumption in period t, w_t is the stock of wealth the individual gathered in period t – made available in period t+1[vi]. Theoretically, such type of "reduced" form arguments are suggested by Bellman's equation formulations of standard accumulated discounted felicity functions[vii] – yet, these imply a special recursive structure of today's wealth evaluation which on the one hand, we leave free, and, on the other, we do not make correspondence to. Rather, a weight of future consumption is embedded in preferences over w_t.

Assume a simple economy: only capital, k_t, can constitute wealth. At each point in time, the representative consumer-producer must decide whether to produce investment goods, i_t, adding to his pre-existing capital stock, or consumption goods, c_t, exhausted in the period, which are homogeneously generated by a CRS production function, implying an average labor product one denoted by $f(k_t)$, with $f(0) = 0$ and$f_k(k_t) > 0$ around the relevant range of k_t:

$$c_t + i_t = f(k_{t-1}) \quad (2)$$

Each unit of capital depreciates at rate d per period. Wealth will evolve according to:

$$k_t = k_{t-1} + i_t - d\, k_{t-1} \quad (3)$$

Hence, at each point in time, *given* a level k_{t-1}, the representative agent's problem – assuming that the utility function is immutable, so that $U_t(c_t, k_t) = U(c_t, k_t)$ for all t - is:

$$\underset{c_t, k_t}{Max}\, U(c_t, k_t) \quad (4)$$

$$\text{s.t:} \quad k_t = (1 - d)\, k_{t-1} + f(k_{t-1}) - c_t \quad (5)$$

$$\text{Given } k_{t-1}$$

or in lagrangean form:

$$\underset{c_t, k_t, \lambda_t}{Max}\, L(c_t, k_t, \lambda_t) = U(c_t, k_t) + \lambda_t\, [k_t - (1 - d)\, k_{t-1} - f(k_{t-1}) + c_t] \quad (6)$$

F.O.C., along with the restriction, require:

$$\frac{\partial L}{\partial c_t} = U_c(c_t, k_t) + \lambda_t = 0 \quad (7)$$

$$\frac{\partial L}{\partial k_t} = U_k(c_t, k_t) + \lambda_t = 0 \quad (8)$$

from where

$$U_c(c_t, k_t) = U_k(c_t, k_t) \quad (9)$$

A first implication is therefore that in each period the consumer is going to equate marginal utility from consumption to that he derives from wealth available for next period income-product generation. At the empirical level, (9) suggests a relation – an “income-expansion” path - between consumption in a period and the lead stock of wealth: past consumption choices condition more closely the current level of attained wealth – and not the other way around...

Another is that if U(c, k) is homothetic, condition c_t and k_t will move at the same proportional change rate along any optimal path – i.e., c_t / k_t is kept constant. For instance:

1) A Cobb-Douglas utility function, $U(c_t, k_t)$ = A c_t□k_t□, would generate: $c_t = \frac{\alpha}{\beta} k_t$. c_t / k_t increases with □ and decreases with □.

2) A CES utility function, $U(c_t, k_t) = A\left[a_1 c_t^{\rho} + a_2 k_t^{\rho}\right]^{\frac{\mu}{\rho}}$ with a_1, a_2> 0, $a_1 + a_2 = 1$, □≤ 1, would imply: $c_t = \left(\frac{a_1}{a_2}\right)^{\frac{1}{1-\rho}} k_t$ – where, as is well-known, $\frac{1}{1-\rho}$ = □ corresponds to the elasticity of substitution between the two arguments. c_t / k_t increases with a_1 and decreases with a_2; it increases (decreases) with □ provided $\frac{a_1}{a_2}$ > (<) 1.

For S.O.C. of the problem to hold, $U[c_t, (1 - d) k_{t-1} + f(k_{t-1}) - c_t]$ should be concave [viii] in c_t, or $U_{cc} + U_{kk} - 2U_{ck} < 0$.

3. Short-Run Dynamics and Steady-State Properties

Along (9):

$$\frac{\partial c_t}{\partial k_t} = \frac{U_{kk} - U_{ck}}{U_{cc} - U_{ck}} = \frac{\dfrac{U_{kk} - U_{ck}}{U_k}}{\dfrac{U_{cc} - U_{ck}}{U_c}} \qquad (10)$$

$\frac{U_{kk} - U_{ck}}{U_{cc} - U_{ck}}$ is expected to be larger than zero. It reflects how consumption is exchanged for capital along any optimal path, i.e., maintaining equality between the marginal utility with respect to capital and that to consumption – keeping the marginal rate of substitution between consumption and capital fixed and equal to 1. It has a similar status to a discount rate – the rate of time preference - in standard discounted utility models: a unit of capital available at the end of the period would be exchangeable or equivalently evaluated to perpetual future consumption flows at that rate, so that rate would be the trade-off with today’s consumption, c_t we should be measuring by $\frac{\partial c_t}{\partial k_t}$ over an optimal path.

Under homothetic utility functions, $\frac{\partial c_t}{\partial k_t}$ is expected to be constant – once it is evaluated at (9); using the previous examples: for the Cobb-Douglas, $\frac{\partial c_t}{\partial k_t} = \frac{\alpha}{\beta}$; for the CES, $\frac{\partial c_t}{\partial k_t} = \left(\frac{a_1}{a_2}\right)^{\sigma}$.

Additively separable utility functions will imply $U_{ck} = 0$, and $\frac{\partial c_t}{\partial k_t}$ is just the ratio between the concavity (or absolute risk-aversion) of U in k to that in c.

An alternative definition of the rate of time preference would be $\frac{U_c(c_{t+1},k_{t+1})[1-d+f_k(k_{t+1})]}{U_c(c_t,k_t)}$ - 1 = $\frac{U_k(c_{t+1},k_{t+1})[1-d+f_k(k_{t+1})]}{U_k(c_t,k_t)}$ - 1. Such ratio would be suggested by the traditional F.O.C. of the Ramsey problem. With a minor adjustment:

$$\frac{U_c(c_{t+1},k_{t+1})[1-d+f_k(k_t)]}{U_c(c_t,k_t)} - 1 = \frac{U_c(c_{t+1},k_{t+1})[1-d+f_k(k_t)]}{U_k(c_t,k_t)} - 1 \qquad (11)$$

would measure the relation between the marginal contribution of today's unit of capital (of the potentially consumable input) for tomorrow's utility – internalizing that $U(c_{t+1}, k_{t+1}) = U[(1-d)k_t + f(k_t) - k_{t+1}, k_{t+1}]$ - relative to the immediate one, minus 1. And, then, in steady-states coinciding with $f_k - d$. One could say that if this definition measures how utility is implicitly evaluated, the other affers it in terms of the consumption capital trade-off. We shall prefer the former definition.

. The equation driving capital dynamics is (5) obeying (9),

$$\frac{\partial k_t}{\partial k_{t-1}} = (1-d) + f_k(k_{t-1}) - \frac{\partial c_t}{\partial k_t}\frac{\partial k_t}{\partial k_{t-1}} = [1-d+f_k(k_{t-1})] / \left(1+\frac{\partial c_t}{\partial k_t}\right)$$

Replacing (10) in the previous expression and solving for

$$\frac{\partial k_t}{\partial k_{t-1}} = [1-d+f_k(k_{t-1})]\,\frac{U_{cc}-U_{ck}}{U_{cc}+U_{kk}-2U_{ck}} \qquad (12)$$

The dynamics of the system can now be studied with reference to the properties of (12). $\frac{\partial k_t}{\partial k_{t-1}}$ is expected to be positive provided U is concave in both arguments. $\frac{\partial k_t}{\partial k_{t-1}} < 1$ and the solution will be stable iff (around the steady-state)

$$f_k(k_{t-1}) - d < \frac{U_{kk}-U_{ck}}{U_{cc}-U_{ck}} = \frac{\partial c_t}{\partial k_t} \qquad (13)$$

Otherwise, it will be unstable. Stability requires that the marginal product of capital, deducted of the depreciation rate (coinciding, in the steady-state, with our second alternative for the definition of the rate of time preference), be smaller than the pseudo-discount rate.

. Being stable, the system will converge to the solution for which $k_t = k_{t-1}$, that is, using (5):

$$c_t = f(k_t) - d\, k_t = f(k_{t-1}) - d\, k_{t-1} \qquad (14)$$

positively sloped while $f_k(k_t) > d$ – certainly for low levels of k under diminishing marginal returns -, and obey (9): k* will be such that

$$U_c[f(k^*) - d\, k^*, k^*] = U_k[f(k^*) - d\, k^*, k^*]$$
(15)

(9) establishes an immediate “saddle-path” trajectory for contemporaneous consumption and capital to follow. The c_t on such path is reached from, for given k_{t-1}:

$$U_c[c_t, (1-d)\, k_{t-1} + f(k_{t-1}) - c_t] = U_k[c_t, (1-d)\, k_{t-1} + f(k_{t-1}) - c_t]$$
(16)

originating a slope

$$\frac{\partial c_t}{\partial k_{t-1}} = [1 - d + f_k(k_{t-1})]\ \frac{U_{kk} - U_{ck}}{U_{cc} + U_{kk} - 2U_{ck}} = [1 - d + f_k(k_{t-1}]\ \frac{1}{1 + \frac{1}{\frac{\partial c_t}{\partial k_t}}}$$
(17)

We can plot the implicit function (16) in space (k_{t-1}, c_t) – below on Fig. 1. Under stability, it will have a smaller slope than the saddle-path (9) evaluated at the lag – i.e., in co-ordinates (k_{t-1}, c_{t-1}), which is also plotted. We plot phaseline (14) – for the lag [ix] - as well – above it k_{t-1} is decreasing: $k_t - k_{t-1} = f(k_{t-1}) - d\, k_{t-1} - c_t < 0$. Below (14), k_{t-1} is rising. (14) has slope, $\frac{\partial c_t}{\partial k_{t-1}} = f_k(k_{t-1}) - d$; it will be smaller than that of (16) provided that (13) holds, i.e., that the system is stable, - and then also smaller than the slope of $\frac{\partial c_{t-1}}{\partial k_{t-1}}$ from (9).

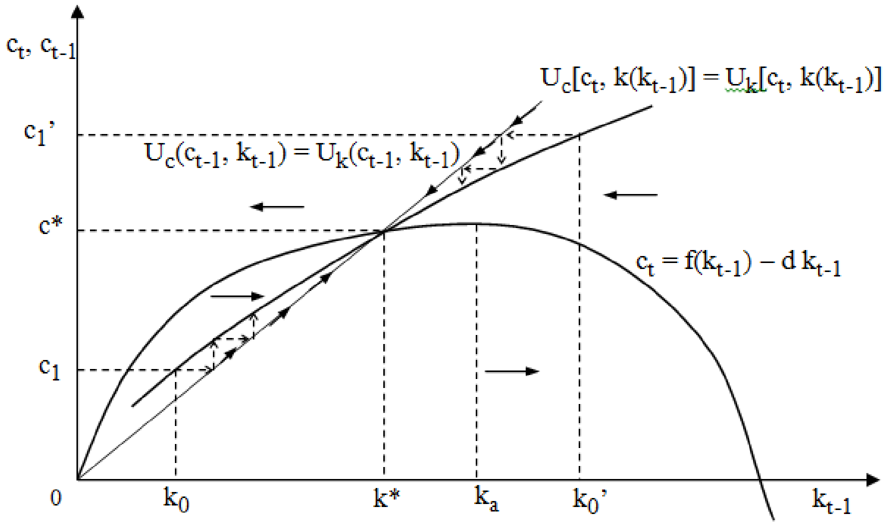

Fig. 1

If we start at a point like k_0 (k_0') consumption will be c_1 (c_1') on line given by (16), and k_1 (k_1') then is read over line (9): we follow the ascending (descendent) steps signaled in the Figure.

Under instability, (16) should have a smaller slope than (14) around the steady-state, and then also (9) would have a smaller slope than (16). The path would be divergent from the steady-state – where both lines meet -, but nevertheless fluctuate (within the space) between (new) lines (16) and (9).

Apparently, the saddle-path properties would resemble those of the Ramsey's problem – see Azariadis, p. 74, for example.

We did not find – unlike in the neoclassical framework – any reason why k_a, the point for which function (14) exhibits a maximum and, therefore, $f_k(k_a) = d$, should be larger than k*. Apparently, then, it is possible that $f_k(k^*) < d$. That might not be the case if we had postulated, instead of (5), a state equation $k_t = (1 - d)\, k_{t-1} + f(k_t) - c_t$, allowing k_t to be immediately available as a production input; not only that would not seem so realistic, as it would render manipulations somewhat more tedious.

. One can show, using (15), that

$$\frac{\partial k^*}{\partial d} = \frac{(U_{cc} - U_{ck})k^*}{(U_{cc} - U_{ck})(f_k - d) + U_{ck} - U_{kk}} = \frac{k^*}{f_k - d + \dfrac{U_{ck} - U_{kk}}{U_{cc} - U_{ck}}} = \frac{k^*}{f_k - d - \dfrac{\partial c_t}{\partial k_t}} \qquad (18)$$

If the system is stable, $\frac{\partial k^*}{\partial d} < 0$.

. Let us assess the likelihood of stability by an example. Assume an homothetic utility function and that $c_t = a\, k_t$, where a is a constant; then, (14) implies $f(k^*) = (a + d)\, k^*$. From (13), stability requires that $f_k(k^*) < a + d$; with the previous, that k*

$f_k(k^*)<f(k^*)$. With CRS, $f(k) - k\,f_k(k)$ equals the marginal product of labor which will be positive. Hence the steady-state will necessarily be stable. In such case, capital dynamics are completely described by $k_t = \frac{(1-d)k_{t-1} + f(k_{t-1})}{1+a}$, positively sloped - once $\frac{\partial k_t}{\partial k_{t-1}} = \frac{1-d+f_k(k_{t-1})}{1+a} > 0$ and $1 - d > 0$ – and concave (in k_t) iff $f_{kk}(k) < 0$ – crossing the 45° line at k^* in space (k_{t-1}, k_t):

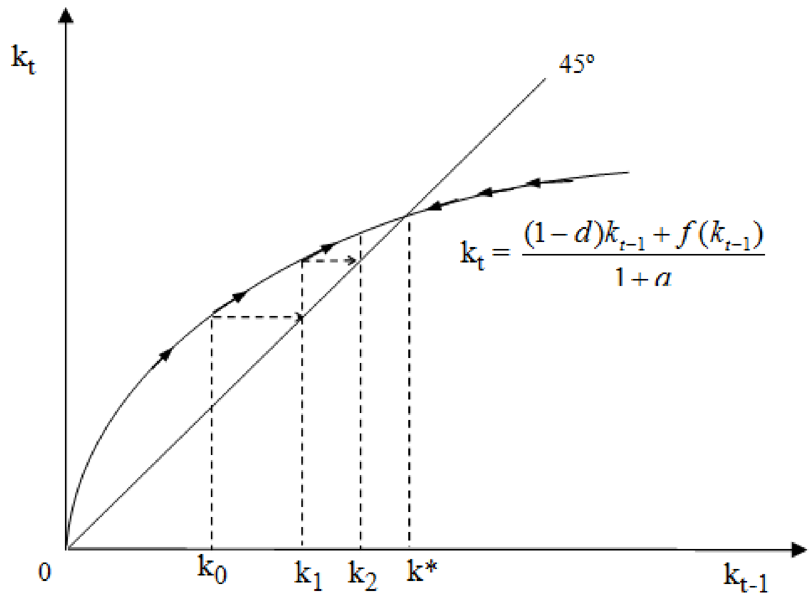

Fig. 2

With those preferences:

$$\frac{\partial k^*}{\partial a} = \frac{k^*}{f_k(k^*) - d - a} \tag{19}$$

Given that stability holds, $\frac{\partial k^*}{\partial a} < 0$: the optimal k^* will decrease with the pseudo-rate of time preference.

. Population growth - at an exogenous constant rate n, i.e., $L_t = (1 + n)\,L_{t-1}$- would not imply any qualitative change to the previous model, provided we keep considering k as the capital-labor ratio, or capital stock per capita: $k_{t-1} = \frac{K_{t-1}}{L_t}$; as in the Solow-Swan model [x], the left hand-side of the capital equation (5) becomes:

$$(1 + n)\,k_t = (1 - d)\,k_{t-1} + f(k_{t-1}) - c_t \tag{20}$$

Then (9) becomes

$$U_c(c_t, k_t) = U_k(c_t, k_t)/(1 + n) \tag{21}$$

The marginal rate of substitution between consumption and capital is now going to be kept at level (1 + n) - $\frac{\partial c_t}{\partial k_t}$ is now $\frac{U_{kk}-(1+n)U_{ck}}{(1+n)U_{cc}-U_{ck}}$. At any level of k_{t-1}, line (9) - the saddle-path, depicted in Fig. 1 - will most likely go up (to (21)): over (21), $\frac{\partial c_t}{\partial n} = \frac{U_c}{U_{ck}-(1+n)U_{cc}}$ and (by SOC) probably positive. For $k_t = k_{t-1}$, $c_t = f(k_{t-1}) - (n + d)\, k_{t-1}$ – line (14) also in Fig. 1 - lowers with n at any level of k_{t-1} -, and in the steady-state:

$$(1 + n)\, U_c[f(k^*) - (d + n)\, k^*, k^*] = U_k[f(k^*) - (d + n)\, k^*, k^*] \tag{22}$$

Now:

$$\frac{\partial k^*}{\partial n} = \frac{[U_{cc}(1+n)-U_{ck}]k^*-U_c}{[U_{cc}(1+n)-U_{ck}](f_k-d-n)+(1+n)U_{ck}-U_{kk}} =$$

$$\frac{k^*-\dfrac{U_c}{U_{cc}(1+n)-U_{ck}}}{f_k-d-n+\dfrac{(1+n)U_{ck}-U_{kk}}{U_{cc}(1+n)-U_{ck}}} \tag{23}$$

For stability – the denominator will be negative -, and for S.O.C. to hold – which suggests that the second term of the numerator will likely be positive -, the optimal capital-labor ratio will decline with n.

It is easily deducted that for homothetic preferences of the CES form the steady-state savings rate 1 – c*/f(k*), because c* = f(k*) – (d + n) k* = a k* where a = $\left(\frac{a_1}{a_2}\right)^{\sigma}(1+n)^{\sigma}$ is a constant, is equal to:

$$s^* = \frac{d+n}{a+d+n} \tag{24}$$

It decreases with a, the "rate of time preference", with $\left(\frac{a_1}{a_2}\right)^{\sigma}$, it increases with d and, provided (1 + n) >□ (d + n), with n - as one encounters in specific cases of the neoclassical model, using say Cobb-Douglas technology and constant elasticity felicity function [xi]. Unlike in these, it will be independent of technology features.

4. Free Market Equilibrium

A decentralized market equilibrium can easily support the previous problem, provided the firms' production function F(K, L) is CRS. Being wages w_t and the interest rate r_t (net of depreciation), firms will maximize profits so as to equate the first to the marginal product of labor, $f(k_{t-1}) - k_{t-1} f_k(k_{t-1})$, the second plus depreciation to that of capital, $f_k(k_{t-1})$, and the economy will follow that same described path with no need for intervention. If agents care for their off-springs – sharing the capital stock -, population growth will not alter the conclusion.

Even decreasing returns to scale – with each individual owning his own production plant and using his own equipment according to $f(k_{t-1})$ – would revert to the previous solution and be efficient, provided there is no population growth.

5. Technical Progress and Human Capital

Technical progress, may generate explosive paths. We shall analyze under which circumstances it may generate stable growth rates. We consider two scenarios: one, in which technical progress is exogenous. Another – in the Uzawa (1965) - Lucas (1988) tradition -, in which it is the product of applied resources to a second sector, that requires but qualified labor to accumulate knowledge or human capital stock, also used in the production of the other goods.

. Admit, then, that the production function is CRS of the type:

$$F(K_{t-1}, A_{t-1} L_t) = A_{t-1} L_t F(\frac{k_{t-1}}{A_{t-1}}, 1) = A_{t-1} L_t f(\frac{k_{t-1}}{A_{t-1}}) \quad (25)$$

A_{t-1} is an efficiency factor affecting labor – labor-augmenting, Harrod-neutral technical progress -, exogenously growing at proportional rate x:

$$A_t = (1 + x) A_{t-1} \quad (26)$$

Then one can convert (20) to:

$$(1 + n) \frac{k_t}{A_{t-1}} = (1 - d) \frac{k_{t-1}}{A_{t-1}} + f(\frac{k_{t-1}}{A_{t-1}}) - \frac{c_t}{A_{t-1}} \quad (27)$$

or

$$(1 + n)(1 + x) \frac{k_t}{A_t} = (1 - d) \frac{k_{t-1}}{A_{t-1}} + f(\frac{k_{t-1}}{A_{t-1}}) - \frac{c_t}{A_t}(1 + x) \quad (28)$$

Provided that U(c, k) is homothetic, condition (21) allows for c_t and k_t to move at the same proportional rate along any optimal path, and it will also be true that:

$$(1 + n) U_c(\frac{c_t}{A_t}, \frac{k_t}{A_t}) = U_k(\frac{c_t}{A_t}, \frac{k_t}{A_t}) \quad (29)$$

Then the problem is stated in such a way that $\frac{k_t}{A_t} = \hat{k}_t$ and $\frac{c_t}{A_t} = \hat{c}_t$ enjoy the same properties as k_t and c_t in the previous model: there will be a steady state *level* $\hat{k}*$ and $\hat{c}*$ that will be stable under similar requirements as before. It involves – as it does for the intertemporal utility function, neoclassical, case - a balanced-growth path for c_t and k_t, moving at the proportional rate x per period, at which A_t grows as well. The steady-state adjusted capital-labor ratio will be such that $\hat{k}_t = \hat{k}_{t-1}$ and we can re-arrange (28) to:

$$\hat{c}_t = [f(\hat{k}_t) - (d + n + x + n x) \hat{k}_t] / (1 + x) \quad (30)$$

In Fig. 1, the line – equivalent to (14) before - will descend with x, and therefore, with stability, the steady-state $\hat{c}*$ and $\hat{k}*$ will decrease with it, once (29) is invariant to x (as long as the latter is positively sloped).

Under homothetic preferences, the steady-state savings rate $1 - c_t/f(k_t) = 1 - (1+x)\,\hat{c}*/f(\hat{k}*)$, because $\hat{c}* = [f(\hat{k}*) - (d + n + x + n\,x)\hat{k}*]/(1 + x) = a\,\hat{k}*$ where a is a constant but dependent on n, is equal to:

$$s^* = \frac{d+n+x(1+n)}{a+d+n+x(1+a+n)} = \frac{1}{1+\dfrac{a(1+x)}{d+n+x(1+n)}} \tag{31}$$

It will increase with x (provided d < 1, which is expected).

. We are going to allow human capital h, to be included in the individual's utility function: on the one hand, it accrues to the individual's productivity potential. On the other, it carries over earnings ability to future periods. The individual can split his time between studying – creating h according to per period function g(.) using only his time endowment, normalized at 1 per period – or producing, l_t.

At each point in time, the planner solves the representative agent's problem:

$$\underset{c_t, k_t, h_t, l_t}{Max}\; U(c_t, k_t, h_t) \tag{32}$$

$$\text{s.t:} \qquad k_t = (1-d)\,k_{t-1} + f(k_{t-1}, h_{t-1}\,l_t) - c_t \tag{33}$$

$$h_t = (1-e)\,h_{t-1} + g[h_{t-1}\,(1 - l_t)] \tag{34}$$

Given k_{t-1} and h_{t-1}

or in lagrangean form:

$$\underset{c_t, l_t, k_t, \lambda_t, \nu_t}{Max}\; L(c_t, l_t, k_t, h_t, \lambda_t, \nu_t) = U(c_t, k_t, h_t) + \lambda_t\,[k_t - (1-d)\,k_{t-1} - f(k_{t-1}, h_{t-1}\,l_t) + c_t] + \tag{35}$$

$$+\; \nu_t\,\{h_t - (1-e)\,h_{t-1} - g[h_{t-1}\,(1 - l_t)]\}$$

F.O.C. require:

$$\frac{\partial L}{\partial c_t} = U_c(c_t, k_t, h_t) + \lambda_t = 0 \tag{36}$$

$$\frac{\partial L}{\partial l_t} = -\lambda_t h_{t-1}\,f_2(k_{t-1}, h_{t-1}\,l_t) + \nu_t h_{t-1}\,g'[h_{t-1}\,(1 - l_t)] = 0 \tag{37}$$

$$\frac{\partial L}{\partial k_t} = U_k(c_t, k_t, h_t) + \lambda_t = 0 \tag{38}$$

$$\frac{\partial L}{\partial h_t} = U_h(c_t, k_t, h_t) + \nu_t = 0 \tag{39}$$

from where we derive that:

$$U_c(c_t, k_t, h_t) = U_k(c_t, k_t, h_t) \tag{40}$$

$$U_c(c_t, k_t, h_t)\,f_2(k_{t-1}, h_{t-1}\,l_t) / U_h(c_t, k_t, h_t) = g'[h_{t-1}\,(1 - l_t)] \tag{41}$$

The two equations generate a saddle path system for c_t and l_t – as a function of (contemporaneous) k_t and h_t -, if we replace k_{t-1} and h_{t-1} from the two state equations in (41).

A stable steady-state for k and h may exist, but for some functional forms, a steady growth rate may be compatible with the optimal solution:

. Assume g(.) is linear in the argument: g(z) = b z and therefore g'(z) = b. Then h_t grows at rate b (1 – l_t) – e. Take also U(c_t, k_t, h_t) to be of the CES type (or similar), so that (40) insures that c_t and k_t will grow at the same proportional rate, i.e., c_t = a k_t where a is a constant. That will determine a saddle-path requirement for c_t and k_t.

Then one can re-write condition (33) as:

$$\frac{k_t}{h_t} = [(1-d)\frac{k_{t-1}}{h_{t-1}} + f(\frac{k_{t-1}}{h_{t-1}}, l_t)] / \{[1 + b(1-l_t) - e](1+a)\} \qquad (42)$$

As f(k_{t-1}, h_{t-1} l_t) is CRS in the two arguments its partial derivatives are homogeneous of degree 0 and condition (41) becomes

$$bU_h(c_t,k_t,h_t)/U_c(c_t,k_t,h_t) = f_2(k_{t-1},h_{t-1}\ l_t) = f[k_{t-1}/(l_t h_{t-1}),1] - \frac{k_{t-1}}{h_{t-1}}\frac{1}{l_t} f_k[k_{t-1}/(l_t h_{t-1}),1] = \qquad (43)$$

$$= \frac{1}{l_t}[f(k_{t-1}/h_{t-1}, l_t) - \frac{k_{t-1}}{h_{t-1}} f_k(k_{t-1}/h_{t-1}, l_t)] = f_2(k_{t-1}/h_{t-1}, l_t)$$

As U is CES, U_h(c_t, k_t, h_t) / U_c(c_t, k_t, h_t) = m(c_t/h_t) where m(.) is a function independent of the arguments of U but c_t/h_t, in which it is increasing. Then, because c_t = a k_t, (43) can be written as:

$$b\ m(a\frac{k_t}{h_t}) = f_2(k_{t-1}/h_{t-1}, l_t) \qquad (44)$$

$$b\ m(a[(1-d)\frac{k_{t-1}}{h_{t-1}} + f(\frac{k_{t-1}}{h_{t-1}}, l_t)] / \{[1 + b(1-l_t) - e](1+a)\}) = f_2(\frac{k_{t-1}}{h_{t-1}}, l_t) \qquad (45)$$

(42) and (44) allow us to determine, at each point in time, $\frac{k_t}{h_t} = \hat{k}_t$ and l_t as a function of, solely, $\frac{k_{t-1}}{h_{t-1}} = \hat{k}_{t-1}$ and describe the whole system dynamics. From (45), and as l_t is not a state variable, its path is determined by it,

$$\frac{\partial l_t}{\partial \hat{k}_{t-1}} = (f_{k2}(\hat{k}_{t-1}, l_t) - b\ a\ m'(.)\ [1 - d + f_k(\hat{k}_{t-1}, l_t)] / \{[1 + b(1-l_t) - e](1 + a)\}) /$$

$$/ \{ b\ a\ m'(.)\ (f_2(\hat{k}_{t-1}, l_t) / \{[1 + b(1-l_t) - e](1+a)\} +$$

$+ b (1 + a) [(1 - d) \hat{k}_{t-1} + f(\hat{k}_{t-1}, l_t)] / \{[1 + b (1 - l_t) - e] (1 + a)\}^2) - f_{22}(\hat{k}_{t-1}, l_t)\}$ (46)

A (contemporaneous) saddle-path could be generated replacing instead $\hat{k}_t$ implicit in the state equation (42) in (44). *Given* $\hat{k}_{t-1}$, l_t is on (45); then, $\hat{k}_t$ would be on that "saddle-path".

From (42),

$$\frac{\partial \hat{k}_t}{\partial \hat{k}_{t-1}} = [1 - d + f_k(\hat{k}_{t-1}, l_t)] / \{[1 + b (1 - l_t) - e] (1 + a)\} + (f_2(\hat{k}_{t-1}, l_t) / \{[1 + b (1 - l_t) - e] (1 + a)\} + b (1 + a) [(1 - d) \hat{k}_{t-1} + f(\hat{k}_{t-1}, l_t)] / \{[1 + b (1 - l_t) - e] (1 + a)\}^2) \frac{\partial l_t}{\partial \hat{k}_{t-1}} \quad (47)$$

It will be positive if (but not only if) $\frac{\partial l_t}{\partial \hat{k}_{t-1}} > 0$. The system will be stable provided (47) is smaller than 1 (in absolute value). It will be smaller than 1 iff ((46) is smaller than):

$$\frac{\partial l_t}{\partial \hat{k}_{t-1}} < [\{[b (1 - l_t) - e] (1 + a) + a + d\} - f_k(\hat{k}_{t-1}, l_t)] [1 + b (1 - l_t) - e] / \{f_2(\hat{k}_{t-1}, l_t) [1 + b (1 - l_t) - e] + b [(1 - d) \hat{k}_{t-1} + f(\hat{k}_{t-1}, l_t)] \} \quad (48)$$

For $\hat{k}_t = \hat{k}_{t-1}$, the dynamic equation (42) becomes:

$$\hat{k}_{t-1} = f(\hat{k}_{t-1}, l_t) / \{[b (1 - l_t) - e] (1 + a) + a + d\} \quad (49)$$

which has slope:

$$\frac{\partial l_t}{\partial \hat{k}_{t-1}} = [\{[b (1 - l_t) - e] (1 + a) + a + d\} - f_k(\hat{k}_{t-1}, l_t)] / (f_2(\hat{k}_{t-1}, l_t) + b (1 + a) \hat{k}_{t-1}) \quad (50)$$

With CRS, it will be positive, once $f(\hat{k}_{t-1}, l_t) > \hat{k}_{t-1} f_k(\hat{k}_{t-1}, l_t)$ insuring positive numerator.

We can plot line (49) on space (k_{t-1}, l_t), along with (45), function $l_t = j(\hat{k}_{t-1})$. Above (49) – because the right hand-side of (42) rises with l_t -, $\hat{k}_{t-1} < \hat{k}_t$ and $\hat{k}_t$ is rising: let $g(\hat{k}_{t-1}, l_t)$ denote the right-hand-side of (42), implying $\hat{k}_t - \hat{k}_{t-1} = g(\hat{k}_{t-1}, l_t) - \hat{k}_{t-1} = 0$ over (49); above it, $\hat{k}_t - \hat{k}_{t-1} > 0$ ($\hat{k}_t$ is rising) iff $\hat{k}_{t-1} < g(\hat{k}_{t-1}, l_t)$. Below (49), the opposite occurs.

If the system is stable – (47) has a slope smaller than 1, being positive, and (48) holds -, (49) will have a higher slope, (50), than that of (46) (because the right hand-side of (48) evaluated at (49) is equal to (50)). If (46) > 0, the saddle-path should have a slope between the two around the steady-state; if negative, it should be more negative that (46). Graphically:

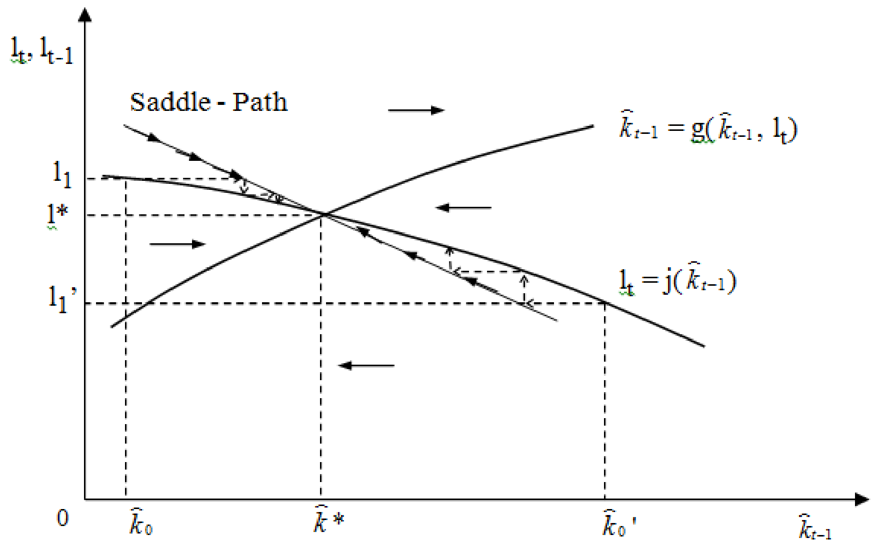

Fig. 3

6. Increasing Returns to Scale and Land

Assume that the aggregate production function is homogeneous of degree □ in the two arguments, aggregate stock of capital, K_t and labor force L_t. IRS (increasing returns to scale) occur for □> 1. As L_t is exogenous, if we can view the production function as CRS in the two arguments, K_{t-1} and $(A_{t-1} L_t)$ of:

$$F(K_{t-1}, A_{t-1} L_t) \quad \text{where} \quad A_{t-1} = L_t^{(□-1)/(1-g)} \tag{51}$$

where g is the degree of homogeneity of $F(K_t, L_t)$ in K_t only – this occurs for a Cobb-Douglas technology, for example -, then we fall under the conditions of exogenous technical progress – and along the stable balanced growth path, k_t and c_t will grow at the same proportional rate as $L_t^{(□-1)/(1-g)}$: $(1 + n)^{(□-1)/(1-g)} – 1$ [xii].

Note, however, that a constant population will allow – unlike in Romer (1986) - for a stable steady-state even with increasing marginal returns to capital, once these are not required – recall (13) – for stability. But of course, they may contend indirectly with the requirement, at least after some level of k_t.

. Admit, on the other extreme that there is also a fixed resource, asset, land, denoted by D, that enters the production function and cannot be changed. Its property is evenly distributed among the population and the representative agent's utility function also depends on it. The aggregate production function is of the type $F(K_{t-1}, L_t, D)$, homogeneous in the two arguments K_t and L_t in such a way that we can write $F(K_{t-1}, L_t, D) = L_t A_{t-1} f(k_{t-1}/A_{t-1}, 1, D)$ where A_{t-1} is a power of L_t. An individual solves:

$$\underset{c_t,k_t,h_t,l_t}{Max}\ U(c_t, k_t, D/L_t)$$

s.t: $(1 + n)\, k_t = (1 – d)\, k_{t-1} + A_{t-1}\, f(k_{t-1}/A_{t-1}, D) - c_t$ (52)

Given k_{t-1}

Then, with population growth, there will be a steady-state balanced growth path in the economy where c_t and k_t grow (or decrease…) at the same proportional rate that A_{t-1} – conditioned by the degree of homogeneity of $F(K_{t-1}, L_t, D)$ in K_{t-1} and L_t only –, as long as along the optimal path, $U_c(c_t, k_t, D/L_t) = U_k(c_t, k_t, D/L_t)$ allows it – say, U(., ., .) is of the CES type in the three arguments. [xiii]

If $F(K_{t-1}, A_{t-1} L_t, D)$ is homogeneous of degree 1 in the three arguments, there will be, in equilibrium with a (gross…) payment of $F_D(K_{t-1}, A_{t-1}L_t, D)$ in real – consumption and/or final product - terms to owners of land per unit of the resource, adjusting $F(K_{t-1}, A_{t-1}L_t, D) = F_k(K_{t-1}, A_{t-1}L_t, D)\ K_{t-1} + F_L(K_{t-1}, A_{t-1}L_t, D)\ L_{t-1} + F_D(K_{t-1}, A_{t-1}L_t, D)\ D$; or, denoting $\widehat{k}_{t-1} = K_{t-1} /A_{t-1}L_t$, $F(K_{t-1}, L_tA_{t-1}, D) / L_tA_{t-1} = F_k[\widehat{k}_{t-1}, D/ (L_tA_{t-1})]\ \widehat{k}_{t-1} + F_L[\widehat{k}_{t-1}, 1, D/ (L_tA_{t-1})] + F_D[\widehat{k}_{t-1}, 1, D/ (L_tA_{t-1})]\ D/ (A_{t-1} L_t)$. In an economy of instantaneous firms, a relative price p^D_t – the price of land in units of either capital, product and, consumption may have to emerge – to account for the fact that they contribute differently to production and consumption and that land does not depreciate – even in the absence of technical progress… Then: $U_D(c_t, k_t, D/L_t) / U_c(c_t, k_t, D/L_t) = U_D(c_t, k_t, D/L_t) / U_k(c_t, k_t, D/L_t) = p^D_t$.

7. Uncertain Wealth

7.1. Additive Uncertainty in Stationary Models

One can hypothesize that the value of the capital stock is a random variable, say added of a noise e_t, translating expectations of future gains from savings applications or expected appreciation or other. Decisions must be made ex-ante, that is, before e_t is observed, and therefore they exhibit no recurrent consequences, or these being independent as long as the external shocks also are. Nevertheless, the disturbance must cause (general) precautionary reaction: now the consumer maximizes expected welfare.

$$\underset{c_t,k_t}{Max}\ E_eU(c_t, k_t + e_t) \quad (53)$$

s.t: $k_t = (1 – d)\, k_{t-1} + f(k_{t-1}) - c_t$ (54)

Given k_{t-1}

or in lagrangean form:

$$\underset{c_t,k_t,\lambda_t}{Max}\ L(c_t, \lambda_t) = E_eU(c_t, k_t + e_t) + \lambda_t\, [k_t - (1 – d)\, k_{t-1} - f(k_{t-1}) + c_t\,]$$

F.O.C., along with the restriction, require:

$$\frac{\partial L}{\partial c_t} = E_eU_c(c_t, k_t + e_t) + \lambda_t = 0 \quad (55)$$

$$\frac{\partial L}{\partial k_t} = E_e U_k(c_t, k_t + e_t) + \lambda_t = 0 \tag{56}$$

from where

$$E_e U_c(c_t, k_t + e_t) = E_x U_k(c_t, k_t + e_t) \tag{57}$$

We can now use Taylor's approximation to expand the marginal utilities around k_t (or we could have expand $U(c_t, k_t + e_t)$ before optimization). Taking the corresponding expected value - assuming the noise has null mean - and denoting the variance of e_t multiplied by 2 (for simplification) by s2: $Var(e_t) = E[e_t^2] = 2\ s2$,

$$U_c(c_t, k_t) + U_{ckk}(c_t, k_t)\ s2 = U_k(c_t, k_t) + U_{kkk}(c_t, k_t)\ s2 \tag{58}$$

or

$$U_c(c_t, k_t) - U_k(c_t, k_t) = [U_{kkk}(c_t, k_t) - U_{ckk}(c_t, k_t)]\ s2$$

For s2 larger than 0, $U_c(c_t, k_t) - U_k(c_t, k_t) > 0$, suggesting a more favorable capital relative to consumption transformation path than the s2=0 case iff $U_{kkk}(c_t, k_t) > U_{ckk}(c_t, k_t)$, that is, if U_{kk} raises more – or – U_{kk}, measuring the concavity of U in k, related to the aversion to a risk added to k, decreases more - per unit increase of capital than per unit rise in consumption.

Now,

$$\frac{\partial c_t}{\partial k_t} = \frac{U_{kk} + U_{kkkk} s2 - U_{ck} - U_{ckkk} s2}{U_{cc} + U_{cckk} s2 - U_{ck} - U_{ckkk} s2} \tag{59}$$

Stability still requires $f_k(k_{t-1}) - d < \frac{\partial c_t}{\partial k_t}$ around the steady-state.

The new consumption path (k_{t-1}, c_t) will satisfy (57) and (54), i.e.:

$$U_c[c_t, (1 - d)\ k_{t-1} + f(k_{t-1}) - c_t] + U_{ckk}[c_t, (1 - d)\ k_{t-1} + f(k_{t-1}) - c_t]\ s2 =$$
$$= U_k[c_t, (1 - d)\ k_{t-1} + f(k_{t-1}) - c_t] + U_{kkk}[c_t, (1 - d)\ k_{t-1} + f(k_{t-1}) - c_t]\ s2 \tag{60}$$

$$\frac{\partial c_t}{\partial s2} = \frac{U_{kkk} - U_{ckk}}{U_{cc} + U_{kk} - 2U_{ck} + U_{cckk} s2 - 2U_{ckkk} s2 + U_{kkkk} s2} \tag{61}$$

Second order conditions require the denominator to be negative – they will be satisfied if not only $U[c_t, (1 - d)\ k_{t-1} + f(k_{t-1}) - c_t]$ but also $U_{kk}[c_t, (1 - d)\ k_{t-1} + f(k_{t-1}) - c_t]$ is concave in c_t. The consumption path will lower with s2 at given k_{t-1} iff $U_{kkk} > U_{ckk}$, i.e., if the marginal utility of consumption is more concave in k than the marginal utility of capital is. Then, as long as that path is positively sloped (which is expected by S.O.C.), – recall Fig. 1 -, k* will rise with uncertainty. Such steady-state value, implying $c^* = f(k^*) - d\ k^*$, requires:

$$U_c[f(k^*) - d\,k^*, k^*] + U_{ckk}[f(k^*) - d\,k^*, k^*]\,s2 = \quad (62)$$
$$= U_k[f(k^*) - d\,k^*, k^*] + U_{kkk}[f(k^*) - d\,k^*, k^*]\,s2$$

or

$$U_c[f(k^*) - d\,k^*, k^*] - U_k[f(k^*) - d\,k^*, k^*] =$$
$$= \{U_{kkk}[f(k^*) - d\,k^*, k^*] - U_{ckk}[f(k^*) - d\,k^*, k^*]\}\,s2$$

The steady-state savings rate, $s^* = 1 - c^*/f(k^*) = d\,k^* / f(k^*)$, will respond to uncertainty according to:

$$\frac{\partial s^*}{\partial s2} = \frac{d[f(k^*) - k^* f_k(k^*)]}{f(k^*)^2}\frac{\partial k^*}{\partial s2} \quad (63)$$

As with CRS $f(k^*) - k^* f_k(k^*)$ equals the, positive, marginal product of labor, s^* will respond to s2 in the same direction as k^* does.

Admit separability between c and k in the utility function such that cross-derivatives are null and we can write

$$U(c_t, k_t) = u^c(c_t) + u^k(k_t) \quad (64)$$

Then (58) becomes:

$$u^c_c(c_t) = u^k_k(k_t) + u^k_{kkk}(k_t)\,s2 \quad (65)$$

and along the saddle-path

$$\frac{\partial c_t}{\partial k_t} = \frac{u^k_{kk} + u^k_{kkkk} s2}{u^c_{cc}} \quad (66)$$

In the steady-state:

$$u^c_c[f(k^*) - d\,k^*] = u^k_k(k^*) + u^k_{kkk}(k^*)\,s2$$

$$\frac{\partial k^*}{\partial s2} = \frac{u^k_{kkk}(k^*)}{u^c_{cc}[f_k(k^*) - d] - u^k_{kk}(k^*) - u^k_{kkkk}(k^*)s2} = \frac{\dfrac{u^k_{kkk}(k^*)}{u^c_{cc}(k^*)}}{f_k(k^*) - d - \dfrac{u^k_{kk}(k^*) + u^k_{kkkk}(k^*)s2}{u^c_{cc}(k^*)}} \quad (67)$$

If the system is stable, the denominator is negative. Then, $\frac{\partial k^*}{\partial s2}$ will be positive and k^* increases with uncertainty iff $\frac{u^k_{kkk}(k^*)}{u^c_{cc}(k^*)} = \frac{-u^k_{kkk}(k^*)}{-u^c_{cc}(k^*)} < 0$: it will if $u^c(c)$ – and U(c,k) - is concave in c and $u^k_k(k)$ – as $U_k(c, k)$ - is convex in k. The condition establishes that if - $\frac{u^k_{kkk}(k^*)}{u^c_{cc}(k^*)}$ - that resembles Kimball's (1990) [xiv] measure of *absolute prudence*, determining how a control variable reacts to added uncertainty in the static context - is positive, k^* will rise with uncertainty.

. Let us consider the reasonable alternative: that optimization behavior is made ex-post and works "deterministically" and that uncertainty only affects production [xv]:

$$\underset{c_t, k_t}{Max}\, U(c_t, k_t)$$

$$\text{s.t:} \quad k_t = (1 - d)(k_{t-1} + e_{t-1}) + f(k_{t-1} + e_{t-1}) - c_t \quad (68)$$

$$\text{Given } k_{t-1} \text{ and } e_{t-1}$$

e_{t-1} is known at time t. Then, obviously, there will not be a "steady-state" for k_t: it will fluctuate according to e_{t-1}, obeying (68) and (9).

Interestingly, (9) and (68) – in general, the current setup - provide a rationale for a co-integrating relation with a structural error-correction mechanism.

Using Taylor expansion on the state equation:

$$k_t = (1 - d)(k_{t-1} + e_{t-1}) + f(k_{t-1}) + f_k(k_{t-1})\, e_{t-1} + \frac{1}{2} f_{kk}(k_{t-1})\, e_{t-1}^2 - c_t \quad (69)$$

From F.O.C, equality between marginal utility of consumption and capital will be satisfied. We could inspect the effect on the "expected" phaseline of a rise in s2, but we would not account for the simultaneous determination of k_t and c_t. Rather, we must consider that the later responds to k_{t-1} according to:

$$U_c[c_t, (1 - d)(k_{t-1} + e_{t-1}) + f(k_{t-1} + e_{t-1}) - c_t] = \quad (70)$$
$$= U_k[c_t, (1 - d)(k_{t-1} + e_{t-1}) + f(k_{t-1} + e_{t-1}) - c_t]$$

(70) establishes a relation $c_t = c(k_{t-1} + e_{t-1})$ identical to that without uncertainty. Expanding around k_{t-1},

$$c_t = c(k_{t-1} + e_{t-1}) = c(k_{t-1}) + \frac{\partial c_t}{\partial k_{t-1}}\, e_{t-1} + \frac{d^2 c_t}{dk_{t-1}^{\,2}}\, e_{t-1}^2 / 2$$

We can write then that:

$$k_t = (1 - d)(k_{t-1} + e_{t-1}) + f(k_{t-1} + e_{t-1}) - c(k_{t-1} + e_{t-1}) = \quad (71)$$
$$= (1 - d)(k_{t-1} + e_{t-1}) + f(k_{t-1}) + f_k(k_{t-1})\, e_{t-1} + \frac{1}{2} f_{kk}(k_{t-1})\, e_{t-1}^2 - c(k_{t-1}) - \frac{\partial c_t}{\partial k_{t-1}}\, e_{t-1} - \frac{1}{2}\frac{d^2 c_t}{dk_{t-1}^{\,2}}\, e_{t-1}^2$$

Let again $Var(e_t) = E[e_t^2] = 2\ s2$. The expected value of k_t at time t is, therefore:

$$E[k_t] = (1 - d)\, k_{t-1} + f(k_{t-1}) + f_{kk}(k_{t-1})\, s2 - c(k_{t-1}) - \frac{d^2 c_t}{dk_{t-1}^{\,2}}\, s2 \quad (72)$$

Stability requires $\frac{\partial k_t}{\partial k_{t-1}}$ to be between –1 and 1. Assume that $\frac{d^2 c_t}{dk_{t-1}^2}$ is negligible. Then:

$$\frac{\partial k_t}{\partial k_{t-1}} = (1-d) + f_k(k_{t-1}) + f_{kkk}(k_{t-1})\, s2\ ] / (1 + \frac{\partial c_t}{\partial k_t})$$

where $\frac{\partial c_t}{\partial k_t}$ comes from the equality between marginal utility of capital and consumption. Admit, for example, homothetic preferences such that $c_t = a\, k_t$. Then:

$$k_t = [(1-d)\, k_{t-1} + f(k_{t-1}) + f_k(k_{t-1})\, e_{t-1} + \frac{1}{2} f_{kk}(k_{t-1})\, e_{t-1}^2] / (1 + a) \quad (73)$$

$$\frac{\partial k_t}{\partial k_{t-1}} = [(1-d) + f_k(k_{t-1}) + f_{kk}(k_{t-1})\, e_{t-1} + \frac{1}{2} f_{kkk}(k_{t-1})\, e_{t-1}^2] / (1 + a) \quad (74)$$

On average, we can expect stability iff

$$s2 f_{kkk}(k_{t-1}) < a + d - f_k(k_{t-1})$$

If $f_{kkk}(k_{t-1}) > 0$ – a plausible assumption -, that requires a low volatility of capital value or productive potential – a low s2. In other words, even if stability were guaranteed under deterministic conditions, if $f_{kkk}(k_{t-1}) > 0$, it is no longer so.

Consider expectations of (73). For a steady-state level of capital, k*:

$$f(k^*) + f_{kk}(k^*)\, s2 = (a + d)\, k^* \quad (75)$$

$$\frac{\partial k^*}{\partial s2} = \frac{f_{kk}(k^*)}{a + d - f_k(k^*) - f_{kkk}(k^*) s2} \quad (76)$$

With stability, diminishing marginal returns to capital - $f_{kk}(k^*) < 0$ – imply that k* decreases with uncertainty (and also c* if the saddle-path is positively sloped, which is expected by SOC).

The steady-state savings rate, $s^* = 1 - c^*/f(k^*) = d\, k^* / [f(k^*) + f_{kk}(k^*)\, s2]$, will be:

$$s^* = \frac{d}{a + d} \quad (77)$$

It will be invariant to uncertainty.

. A juxtaposition of the two effects would be realistic: that the agent solves:

$$\underset{c_t, k_t}{Max}\ E_e U(c_t, k_t + e_t)$$

$$\text{s.t: } k_t = (1-d)(k_{t-1} + e_{t-1}) + f(k_{t-1} + e_{t-1}) - c_t$$

Given k_{t-1} and e_{t-1}

The previous decomposition allows us to distinguish the utility and technology channels through which uncertainty – dispersion of tastes - affects the equilibrium.

7.2. Multiplicative Uncertainty

.Suppose:

$$\max_{c_t, k_t} E_e U[c_t, k_t (1 + e_t)] \quad (78)$$

s.t: $k_t = (1 – d) k_{t-1} + f(k_{t-1}) - c_t$ (79)

Given k_{t-1}

Admit further separability of the utility function so that we can write: $U[c_t, k_t (1 + e_t)] = u^c(c_t) + u^k[k_t (1 + e_t)]$ so that $E_e U[c_t, k_t (1 + e_t)]$ □ $u^c(c_t) + u^k(k_t) + u^k_{kk}(k_t) k^2 s2$. Then:

$$u^c_c(c_t) = u^k_k(k_t) + [u^k_{kkk}(k_t) k_t^2 + 2 u^k_{kk}(k_t) k_t] s2$$

(80)

and along the saddle-path:

$$\frac{\partial c_t}{\partial k_t} = \frac{u^k_{kk} + (k_t^2 u^k_{kkkk} + 4k_t u^k_{kkk} + 2u^k_{kk})s2}{u^c_{cc}} \quad (81)$$

In the steady-state:

$$u^c_c[f(k^*) - d\,k^*] = u^k_k(k^*) + [u^k_{kkk}(k^*) k^{*2} + 2 u^k_{kk}(k^*) k^*]$$

s2

$$\frac{\partial k^*}{\partial s2} = \frac{u^k_{kkk}(k^*)k^{*2} + 2u^k_{kk}(k^*)k^*}{u^c_{cc}[f_k(k^*) - d] - u^k_{kk}(k^*) - [u^k_{kkkk}(k^*) + 2k^* u^k_{kkk}(k^*)]s2} =$$

$$= \frac{\dfrac{u^k_{kkk}(k^*)k^{*2} + 2u^k_{kk}(k^*)k^*}{u^c_{cc}}}{[f_k(k^*) - d] - \dfrac{u^k_{kk}(k^*) + [u^k_{kkkk}(k^*) + 2k^* u^k_{kkk}(k^*)]s2}{u^c_{cc}}} \quad (82)$$

If the system is stable, it will be positive iff $\frac{u^k_{kkk}(k^*)k^{*2} + 2u^k_{kk}(k^*)k^*}{u^c_{cc}} < 0$: it will if $u^c(c)$ – and U(c,k) - is concave in c and $[u^k_{kk}(k) k^2]$ rises with k.

. Finally, let preferences be homothetic and admit

$$k_t = (1 – d) k_{t-1} (1 + e_{t-1}) + f[k_{t-1} (1 + e_{t-1})] - c_t \quad (83)$$

Now

$$k_t = [(1 – d) (k_{t-1} + e_{t-1}) + f(k_{t-1}) + f_k(k_{t-1}) k_{t-1}e_{t-1} + \frac{1}{2} f_{kk}(k_{t-1}) k_{t-1}^2$$

$e_{t-1}^2] / (1+a)$ (84)

In the expected steady-state

$$f(k^*) + k^{*2} f_{kk}(k^*) s2 = (a + d) k^* \quad (85)$$

$$\frac{\partial k^*}{\partial s2} = \frac{k^{*2} f_{kk}(k^*)}{a + d - f_k(k^*) - [k^{*2} f_{kkk}(k^*) + 2k^* f_{kk}(k^*)]s2} \quad (86)$$

As before, with stability, diminishing marginal returns to capital - $f_{kk}(k^*) < 0$ – insure that k* (and c*) decreases with uncertainty.

. If we are in the presence of exogenous labor-augmenting technical progress, balanced growth would be recovered (at least on average), with system dynamics towards k_t/A_t approaching that of k_t in the current framework. Given that uncertainty factors capital, we do not expect effects of uncertainty on balanced growth rates even if effects remain in steady-sate ratios.

8. Overlapping Optimization: Recursive Structures.

One could argue that the previous problem fails to capture forward-looking intertemporal effects. That may not be so, but we could assume then that k_{t+1} also enters the individual's utility function at time t and that the following period's capital constraint is (therefore) also considered in the current problem. Then the finite-horizon problem – problems - would be [xvi]:

$$\underset{c_t, k_t, k_{t+1}}{Max}\ U(c_t, k_t, k_{t+1}) \quad , \quad t = 1,2,\dots, T \quad (87)$$

$$\text{s.t:} \quad k_t = (1 - d) k_{t-1} + f(k_{t-1}) - c_t \quad , \quad t = 1,2,\dots, T$$

$$k_{t+1} = (1 - d) k_t + f(k_t) - c_{t+1} \quad , \quad t = 1,2,\dots, T\text{-}1 \quad (88)$$

$$\text{Given } k_0, k_{T+1}$$

or in lagrangean form:

$$\underset{c_t, k_t, k_{t+1}, \lambda_t, \nu_t}{Max}\ L(c_t, \lambda_t) = U(c_t, k_t, k_{t+1}) + \lambda_t [k_t - (1 - d) k_{t-1} - f(k_{t-1}) + c_t] + \quad (89)$$

$$+ \nu_t [k_{t+1} - (1 - d) k_t - f(k_t) + c_{t+1}]$$

F.O.C., along with the restrictions, require for t = 1,2,…,T-1 that

$$\frac{\partial L}{\partial c_t} = U_c(c_t, k_t, k_{t+1}) + \lambda_t = 0$$

$$\frac{\partial L}{\partial k_t} = U_{k1}(c_t, k_t, k_{t+1}) + \lambda_t - \nu_t [(1 - d) + f_k(k_t)] = 0$$

$$\frac{\partial L}{\partial k_{t+1}} = U_{k2}(c_t, k_t, k_{t+1}) + \nu_t = 0$$

from where

$$U_c(c_t, k_t, k_{t+1}) = U_{k1}(c_t, k_t, k_{t+1}) + U_{k2}(c_t, k_t, k_{t+1}) [(1 - d) + f_k(k_t)] \ , \ t = 1,2,\dots,T\text{-}1 \quad (90)$$

$$U_c(c_T, k_T, k_{T+1}) = U_{k1}(c_T, k_T, k_{T+1}) \quad (91)$$

There are (T + T) unknowns – (c_t, k_t), t = 1,2,…,T -, and T + T equations – (90) and (91) and the T generic state equations, (88) [xvii]. Therefore, the problem should have a well-defined solution, obeying

$$c_t = c(k_{t-1}, c_{t+1}),\ k_t = k1(k_{t-1}, c_{t+1}),\ k_{t+1} = k2(k_{t-1}, c_{t+1}),\ t = 1,2,\dots T\text{-}1 \qquad (91)$$
$$c_T = cT(k_{T-1}, k_{T+1}),\ \ k_T = kT(k_{T-1}, k_{T+1})$$

The rate of time preference would become:

$$\frac{\partial c_t}{\partial k_t} = \frac{U_{k1k1} + U_{k1k2}(1-d+f_k) + U_{k2} f_{kk} - U_{ck1}}{U_{cc} - U_{ck1} - U_{ck2}(1-d+f_k)} \qquad (92)$$

Again, we could propose as an alternative definition: $\frac{U_c(c_{t+1}, k_{t+1}, k_{t+2})[1-d+f_k(k_{t+1})]}{U_c(c_t, k_t, k_{t+1})}$ - 1 or $\frac{U_c(c_{t+1}, k_{t+1}, k_{t+2})[1-d+f_k(k_t)]}{U_c(c_t, k_t, k_{t+1})}$ - 1.

S.O.C. would require U{c_t, (1 – d) k_{t-1} + f(k_{t-1}) - c_t, (1 – d) [(1 – d) k_{t-1} + f(k_{t-1}) - c_t] + f[(1 – d) k_{t-1} + f(k_{t-1}) - c_t] – c_{t+1}} concave in c_t, i.e., that U_{cc} - U_{k1c} - U_{k2c} (1 – d + f_k) - U_{ck1} + U_{k1k1} + U_{k2k1} (1 – d + f_k) –[U_{ck2} - U_{k1k2} - U_{k2k2} (1 – d + f_k)](1 – d + f_k) + U_{k2} f_{kk}< 0. They are therefore satisfied with decreasing marginal utility with respect to each argument (U_{jj}< 0, all j), positive U_{cj}'s, j = k1, k2, and negative U_{k1k2}.

In infinite horizons, (90) and the state equations define the properties of the optimal path; a boundary – or limiting transversality-like - condition could replace the establishment of k_{T+1}. A steady-state would satisfy:

$$U_c[f(k^*) - d\,k^*, k^*, k^*] = U_{k1}[f(k^*) - d\,k^*, k^*, k^*] + \qquad (94)$$
$$+ U_{k2}[f(k^*) - d\,k^*, k^*, k^*]\,[1 - d + f_k(k^*)]$$

One can study the optimal solution dynamics by analyzing the system (around the steady-state, at least) [xviii]:

$$U_c(c_t, k_t, k_{t+1}) = U_{k1}(c_t, k_t, k_{t+1}) + U_{k2}(c_t, k_t, k_{t+1})\,[1 - d + f_k(k_t)] \qquad (95)$$
$$k_{t+1} = (1 - d)\,k_t + f(k_t) - c_{t+1} \qquad (96)$$

From the two, we can generate:

k_{t+1} = g1(k_t, c_t) (This, immediately from (95)) (97)

c_{t+1} = g2(k_t, c_t) (In our system, g2(k_t, c_t) = (1 – d) k_t + f(k_t) - g1(k_t, c_t)) (98)

The (2x2) Jacobian matrix A = [a_{ij}], would contain:

$$a_{11} = \frac{\partial k_{t+1}}{\partial k_t} = \frac{U_{k1k1} + U_{k1k2}(1-d+f_k) + U_{k2} f_{kk} - U_{ck1}}{U_{ck2} - U_{k1k2} - U_{k2k2}(1-d+f_k)}$$

$$a_{12} = \frac{\partial k_{t+1}}{\partial c_t} = \frac{U_{k1c} + U_{k2c}(1-d+f_k) - U_{cc}}{U_{ck2} - U_{k1k2} - U_{k2k2}(1-d+f_k)}$$

$$a_{21} = \frac{\partial c_{t+1}}{\partial k_t} = [1 - d + f_k(k_t)] - a_{11}$$

$$a_{22} = \frac{\partial c_{t+1}}{\partial c_t} = - a_{12}$$

It has trace T and determinant D, having correspondence with the eigenvalues of A, r_1 and r_2, in such a way that:

$$T = a_{11} + a_{22} = \frac{U_{k1k1} + U_{cc} + (U_{k1k2} - U_{ck2})(1 - d + f_k) + U_{k2} f_{kk} - 2U_{ck1}}{U_{ck2} - U_{k1k2} - U_{k2k2}(1 - d + f_k)} = r_1 + r_2 < 0$$

$$D = a_{11}\, a_{22} - a_{12}\, a_{21} = - [1 - d + f_k(k_t)]\, a_{12} =$$

$$= - [1 - d + f_k(k_t)]\, \frac{U_{k1c} + U_{k2c}(1 - d + f_k) - U_{cc}}{U_{ck2} - U_{k1k2} - U_{k2k2}(1 - d + f_k)} = r_1 r_2 < 0$$

If in moduli one eigenvalue is larger than one and the other is smaller than one, the system is unstable and possesses a saddle-path that converges to the steady-state. The determinant and the trace most likely are negative (if we assume decreasing marginal utility with respect to each argument, positive U_{cj}'s, j = k1, k2, and negative U_{k1k2}); $T^2 - 4D$ is then positive and the two roots are real. As D and T are negative, one eigenvalue is positive and the other is negative. As their sum - T - is negative, numbering the regions in space (T, D) according to Azariadis (1998), p.65-66:

If $D = - [1 - d + f_k(k_t)]\, a_{12} < - 1$, i.e. $f_k(k_t) > d$ for a_{12} larger or equal to 1, the steady-state is:

Case A: a source (unstable) if $D < T - 1$: both have moduli larger than 1 - Region (2).

Case B: a saddle if $D > T - 1$, $D < - (T + 1)$ - for $D < -1$ ($T < 0$), only the first bound is relevant: both eigenvalues are on the same side of –1, different sides of 1: one is in (-1, 1), the other in (1, □)– Region (3).

If $- 1 < D < 0$; as $T < 0$, the steady-state is either

Case C: a saddle if $D > T - 1$, $D < - (T + 1)$ – for $-1 < D < 0$ ($T < 0$), only the second bound is relevant: both eigenvalues are on the same side of –1, different sides of 1: one is in (-1, 1), the other in (1, □)– Region (3).

Case D: a sink (stable) if $D > - (T + 1)$: both eigenvalues fall in (-1,1) – Region (7b).

Case E: a flip (period-doubling) bifurcation (Azariadis, p. 93) if $D = - (T + 1)$.

We can compute:

$$T - 1 = a_{11} - a_{12} - 1$$

$$- (T + 1) = - (a_{11} - a_{12} + 1)$$

As $D = - [1 - d + f_k(k_t)]\, a_{12}$, $D > T - 1$ implies $a_{11} - 1 < - [f_k(k_t) - d]\, a_{12}$. If $f_k(k_t) > d$, this will necessarily occur (because $a_{11} < 0$ and $a_{12} > 0$ are most likely). Then, we rule out case A.

$D < - (T + 1)$ translates to $- [1 - d + f_k(k_t)]\, a_{12} < - (a_{11} - a_{12} + 1)$. We could not prove that the opposite cannot occur, which would discard Case D (and E). But at least $D < - (T + 1)$ would cover – once $T < 0$ – a larger range of possibilities.

Alternatively, we can rely on the simpler analysis of the variables' trends around the functions $k_{t+1} = g1(k_t, c_t)$ and $c_{t+1} = g2(k_t, c_t)$ evaluated at the steady-state, i.e., for $k_{t+1} = k_t$ and $c_{t+1} = c_t$ – the phaselines. We plot the resulting conclusions – valid for linear approximations around the steady-state - in the phase diagrams, Fig. 4 and 5, below:

Taking (97) for steady-state k_t, $k_t = g1(k_t, c_t)$ (or (95)) and evaluating its slope at $k_{t+1} = k_t$ and $c_{t+1} = c_t$, i.e. on:

$$U_c(c_t, k_t, k_t) = U_{k1}(c_t, k_t, k_t) + U_{k2}(c_t, k_t, k_t)\,[1 - d + f_k(k_t)] \qquad (99)$$

We derive:

$$\frac{\partial c_t}{\partial k_t} \square\ \square 1 - a_{11}) / a_{12} > 0 \qquad (100)$$

It is (if T < 0, a justified assumption) positively sloped: above (99), k_t is rising. $k_{t+1} - k_t = g1(k_t, c_t) - k_t$; it will be larger than 0 and k_t is rising iff $g1(k_t, c_t) > k_t$ – once at a given k_t, as $g1_c = a_{12} > 0$, for values of c_t to the right of the line g1 shows larger values (and then, larger than k_t).

Repeating the same exercise for (98), $c_t = g2(c_t, k_t)$ that we can solve for $c_t = g3(k_t)$ - that differs from (14). We have that over it:

$$\frac{\partial c_t}{\partial k_t} = a_{21} / (a_{12} + 1) > 0. \qquad (101)$$

It is also most likely positively sloped: to the right of $c_t = g2(k_t, c_t)$, c_t is increasing. $c_{t+1} - c_t = g2(k_t, c_t) - c_t$; it will be larger than 0 and c_t is rising iff $g2(k_t, c_t) > c_t$ – once at a given c_t, as $g2_k = a_{21} > 0$ (most likely), g2 is larger for larger values of k_t than over the line.

Then either:

$c_t = g2(k_t, c_t)$ has a higher slope than $k_t = g1(k_t, c_t)$ – Fig. 4 – and we have a saddle-path (Case C):

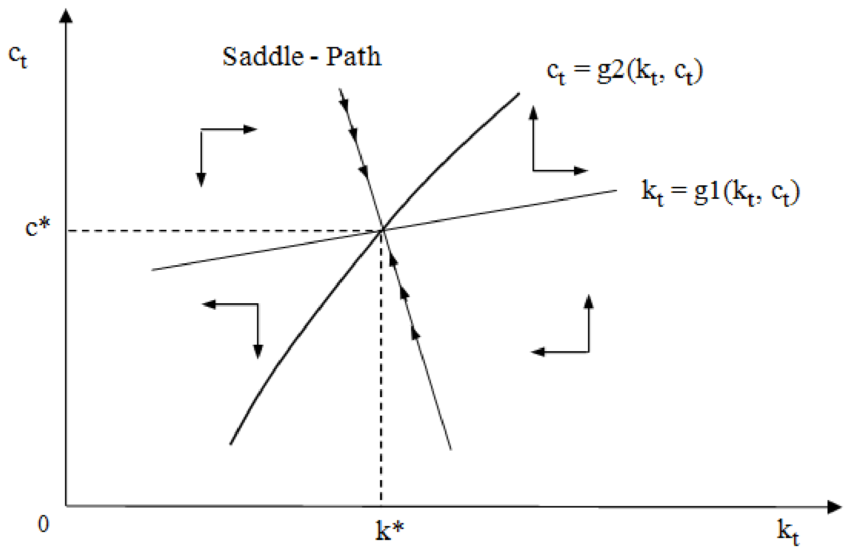

Fig. 4

The arrows point to the steady-state, where the two functions meet, in only two quadrants (the system is not stable) where the saddle-path must lie, exhibiting the pattern shown in the figure. Interestingly, the saddle-path appears negatively sloped – a different pattern from that of Fig.1.

Or $c_t = g2(k_t, c_t)$ has a smaller slope than $k_t = g1(k_t, c_t)$ – depicted in Fig. 5 – and we have a (stable) "sink" steady-state – Case D:

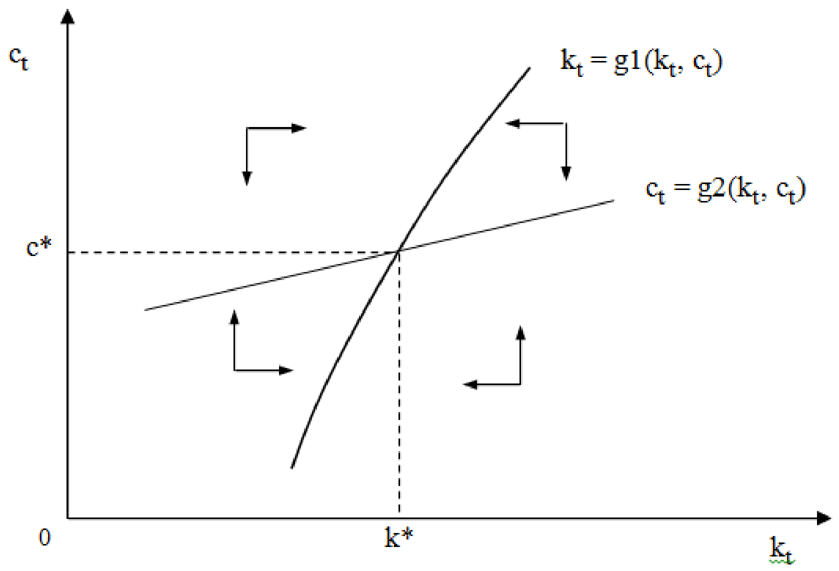

Fig. 5

A final comment on the optimization structure should be added. One could forward an optimization procedure where intertemporal efficiency was also required:

$$\underset{c_t, k_t, M_t}{Max}\ U(c_t, k_t, k_{t+1})$$

$$k_{t+j} = (1 - d)\, k_{t+j-1} + f(k_{t+j-1}) - c_{t+j} \quad , \quad j = 0, 1 \text{ (or 0, 1, 2, 3)} \tag{102}$$

$$U(c_{t+j}, k_{t+j}, k_{t+j-1}) \ \square\ \overline{U}_{t+j,t} \quad , \quad j = 1 \text{ (or 1, 2, 3)} \tag{103}$$

Given k_{t-1}

In lagrangean form:

$$\underset{k_t, k_{t+1}, \xi_t}{Max}\ L(c_t, k_t, k_{t+1}, \zeta_t) = U(c_t, k_t, k_{t+1}) + \lambda_t\, [k_t - (1 - d)\, k_{t-1} - f(k_{t-1}) + c_t] +$$
$$+ \nu_t\, [k_{t+1} - (1 - d)\, k_t - f(k_t) + c_{t+1}] +$$
$$+ \zeta_t\, \{\overline{U}_{t+1,t} - U(c_{t+1}, k_{t+1}, k_{t+2})\} \tag{104}$$

However, at time t, $\overline{U}_{t+j,t}$ is still forthcoming – $\overline{U}_{t+j,t}$ would not even have to equal $\overline{U}_{t+j,t+s}$: the corresponding multiplier should in fact be 0. In fact we are assuming that different people may proceed to subsequent optimization and freeing the analogous constraint that implicitly structures accumulated discounted utility maximizers.

9. Conclusion

We explored the potential of point-wise optimization of individuals' utility functions dependent on consumption and wealth to reproduce the dynamic behavior of the macroeconomy. The framework proved to be capable of generating similar dynamics as traditional and neoclassical real growth models – based on intertemporal utility maximization. Exogenous population growth, technical progress and increasing returns showed similar consequences and stability requirements as "conventional" models do.

With the introduction of human capital, more sophisticated scenarios could be explored. Under the current framework, the inclusion of human capital along with material wealth in the utility function becomes natural. Again, Uzawa's technological setup implied the same dynamic patterns.

One can ask then why would the proposed function should be of use? Firstly, because it does not require an hypothesis of accumulated discounted utility maximand. Rather, an implicit or pseudo-rate of time preference was mathematically deducted, relating the rate of change of consumption with that of capital along an optimal path - homothetic preferences rendering it constant. Eventually, time inconsistency [xix] based on discounting patterns therefore disappears.

Secondly, because it shifts the attention towards and stresses wealth formation – resulting from, caused by, the accumulation of past saving-investment – consumption abstinence – flows. Maybe we should not be looking for a long-run consumption function, but for a wealth or asset (demand) function – dependent on past consumption... Or functions of the two – wealth and lagged consumption - may just be more closely co-integrated.

Thirdly, for its mathematical tractability – even if we foresee that the inclusion of money, bonds, taxes, public goods, multiple assets or market goods – debt, a potential negatively valued argument of the WIU function -, or yet life-cycle labor-leisure choices, natural extensions or applications that we did not pursue here, may complicate it again. That allowed us to make reasonable deductions of matters like the effect of exogenous uncertainty – shocks – on tastes or technology over intertemporal dynamics of economic stocks and flows – reviewing the role of risk-aversion and pace of diminishing marginal returns to capital in growth determination.

Finally, forward-looking dynamics were found to be compatible with capital-in-utility (wealth-in-utility) modeling – sufficing to include the capital stock of two (an extra) future periods in the welfare function, optimization being thus made conditional on future decisions.

Obviously, applications of the same principles to the intertemporal decisions of the firm are in the agenda.

Notes

[i] Cass (1965) and Koopmans (1965). Kurz (1968) includes wealth effects in the felicity function.
[ii] Or status effect models – see Bakshi & Chen (1996) for an example. See also Zou (1998).
[iii] Zou (1995).
[iv] Bakshi & Chen (1996).
[v] Diamond (1965).
[vi] One could postulate as well $U_t(c_t, w_{t+1})$. Provided that in the fundamental dynamic wealth equation(s) below w_t (k_t) is replaced by w_{t+1} (k_{t+1}) for all t and w_{t+1} is determined with c_t, the conclusions would remain.
[vii] See, for example, Romer (1996), Ex. 4.11., p. 192-193. Or Blanchard & Fischer (1989), p.284.
[viii] With quasi-concavity of $U(c_t, k_t)$ in the arguments being sufficient to generate convex indifference contours in space (k_t, c_t) - where for given k_{t-1}, the (budget) constraint (5) is linear with slope –1 - and that F.O.C. imply a maximum.
[ix] Theoretically, if c_{t-1} entered the system as well, the curve to be plotted in a phase diagram, the phaseline, would not be this one – see section 7 below. Given the simple structure of the problem, the argument holds. Also, notice we are plotting c_t against k_{t-1}, which is not usual, but here sensible.
[x] See Azariadis (1998), p. 4, for example.
[xi] See Martins (1989), for example.
[xii] Arrow's (1962) IRS technology generates similar consequences.
[xiii] Under increasing returns, a competitive solution will hardly guarantee a pareto optimal result. Market solutions with externalities are found in Martins (1987) and (1989), for example.
[xiv] See also Martins (2004).
[xv] A generalization of Ramsey's problem with ex-ante uncertainty in production was studied by Brock & Mirman (1972). Of course, anticipation would have more complex effects than ours when intertemporal optimization is considered.
[xvi] We could think that including the second restriction in the former, simpler, problem would render a recursive structure. It does not, once, as we have two lagrange multipliers, with only two controls, we are left with only the two state equations per periodic problem. Hence, k_{t+1} is "conditionally" decided today, and we have to confirm – or specify - our decision on k_{t+1} next period. Also, only k_t and k_{t+1} appear in the current function, for current decision – hence, only two constraints (one for each of them) are relevant.
[xvii] As one restriction always overlaps in two subsequent period problems, decisions over capital are forced to be consistent.
[xviii] Technically, the F.O.C generate now similar dynamic traits as Ramsey's structure – see Azariadis, p. 210, for example.
[xix] See an early reference in Strotz (1955), and Frederick Loewenstein & O'Donoghue (2002) for a recent survey.

References

Aghion, P., & Howitt, P. (1998). *Endogenous Growth Theory*.Cambridge, Massachusetts Institute of Technology.

Arrow, K.J. (1962). The economic implications of learning by doing. *Review of Economic Studies*, 29, 155-73.

Azariadis, C. (1998). *Intertemporal Macroeconomics*.Oxford, Blackwell Publishers.Reprint.

Bakshi, G., & Chen, Z. (1996). The spirit of capitalism and stock market prices. *American Economic Review*, 86(1), 133-157

Barro, R.J., & Sala-i-Martin, X. (1995). *Economic Growth*.New York, McGraw-Hill.

Blanchard, O.J., & Fischer, S. (1989). *Lectures on Macroeconomics*.Cambridge, Massachusetts Institute of Technology.

Brock, W.A., & Mirman, L.J. (1972). Optimal growth and uncertainty: The discounted case. *Journal of Economic Theory*, 4(3), 479-513. doi. 10.1016/0022-0531(72)90135-4

Cass, D. (1965). Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies*, 32(3), 233-40. doi. 10.2307/2295827

Diamond, P. (1965). National debt in a neoclassical growth model. *American Economic Review*, 55(5), 1126-1150.

Frederick, S., Loewenstein, G., & O'Donoghue, T. (2002). Time discounting and time preference: A critical review. *Journal Economic Literature*, 40(2), 351-401. doi. 10.1257/002205102320161311

Kimball, M.S. (1990). Precautionary saving in the small and in the large. *Econometrica*. 58(1), 53-73. doi. 10.2307/2938334

Kurz, M. (1968). Optimal economic growth and wealth effects. *International Economic Review*, 9(3), 348-357. doi. 10.2307/2556231

Lucas, R.E. (1988). On the mechanics of economic development. *Journal of Monetary Economics*, 22(1), 3-42. doi. 10.1016/0304-3932(88)90168-7

Martins, A.P. (1987). *Human Capital and Economic Growth*. Unpublished Columbia University, Ph.D. dissertation.

Martins, A.P. (1989). Increasing Returns and Endogenous Growth: Market Size and Taste for Variety. Working Paper Nº 133. Faculdade de Economia da Universidade Nova de Lisboa.

Martins, A.P. (2004). Uninsurable Risks: Uncertainty in Production, the Value of Information and Price Dispersion. Mimeo.

Ramsey, F.P. (1928). A mathematical theory of saving. *Economic Journal*. 38(152), 543-559. doi. 10.2307/2224098

Romer, D. (1996). *Advanced Macroeconomics*. New York, McGraw-Hill.

Romer, P. (1986). Increasing returns and long run growth. *Journal of Political Economy*, 94(5), 1002-37. doi. 10.1086/261420

Samuelson, P.A. (1958). An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*. 66(6), 467-482. doi. 10.1086/258100

Sidrauski, M. (1967). Rational choice and the patterns of growth in a monetary economy. *American Economic Review*. 57(2), 534-544.

Solow, R. (1956). A Contribution to the Theory of Economic Growth. *Quarterly Journal of Economics*. 70(1), 65-94. doi. 10.2307/1884513

Strotz, R.H. (1955-1956). Myopia ans inconsistency in dynamic utility maximization. *Review of Economic Studies*. 23(3), 165-180. doi. 10.2307/2295722

Swan, T.W. (1956). Economic growth and capital accumulation. *Economic Record*. 32(2), 334-361. doi. 10.1111/j.1475-4932.1956.tb00434.x

Uzawa, H. (1965). Optimal technical change in an aggregative model of economic growth. *International Economic Review*. 6(1), 18-31. doi. 10.2307/2525621

Walsh, C.E. (2003). *Monetary Theory and Policy*.Cambridge, Massachusetts Institute of Technology, 2nd Ed.

Zou, H.-F. (1998). Thespirit of capitalism, social status, money, and accumulation. *Journal of Economics*. 68(3), 219-233. doi. 10.1007/BF01237193

Zou, H.-F. (1995). Thespirit of capitalism and savings behavior. *Journal of Economic Behavior and Organization*. 28(1), 131-143. doi. 10.1016/0167-2681(95)00024-2

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